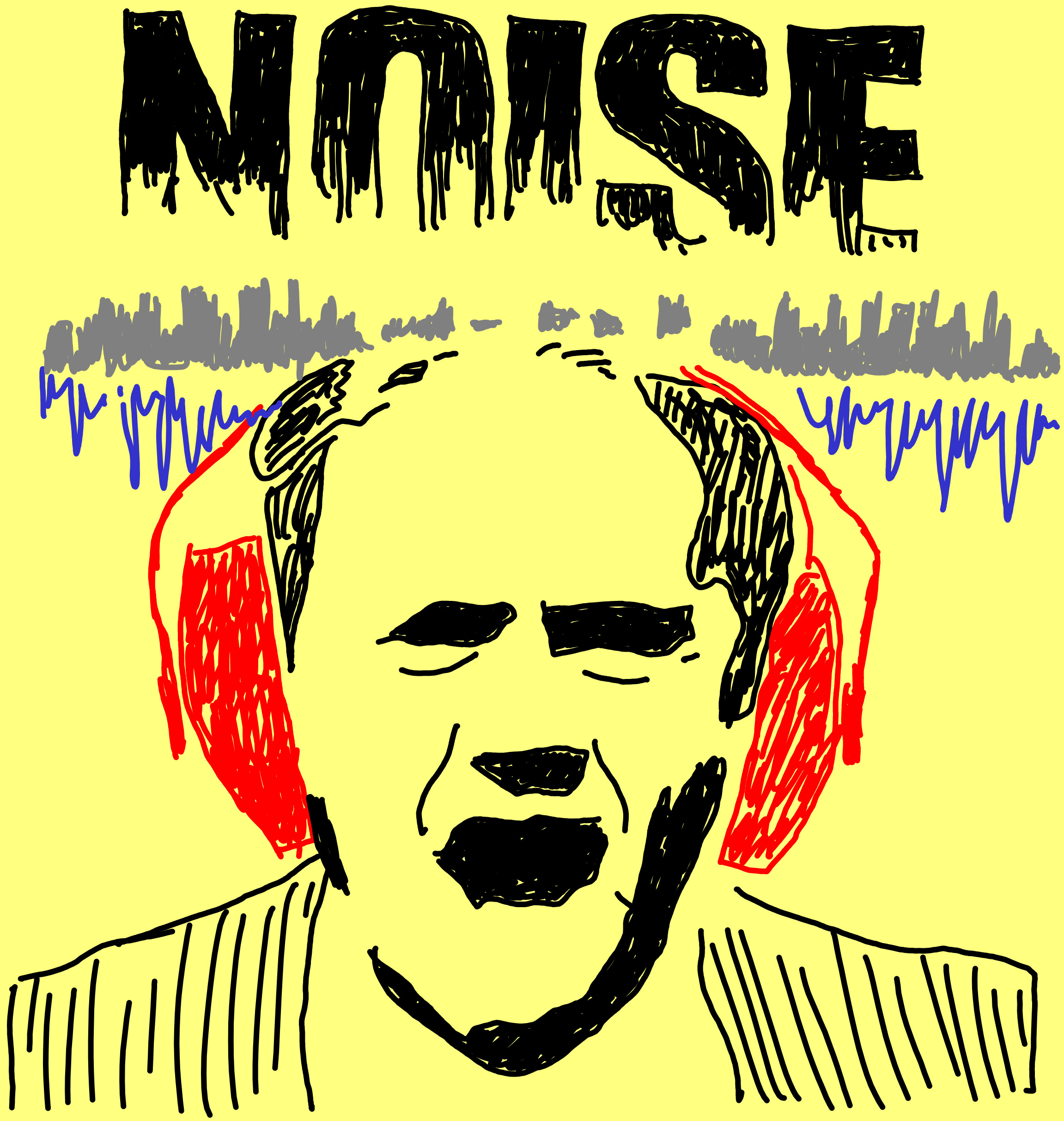
Measuring with Algebra

CTIC '09

Measuring with Algebra

Herbert Edelsbrunner

IST Austria,
Duke University & Geomagic


NOISE

NOISE

. . . IS ALWAYS AND EVERYWHERE.

DOGMA:

measure features and threshold noise

(without changing the data)

I Persistent Homology

II L_p-Stability and Somites

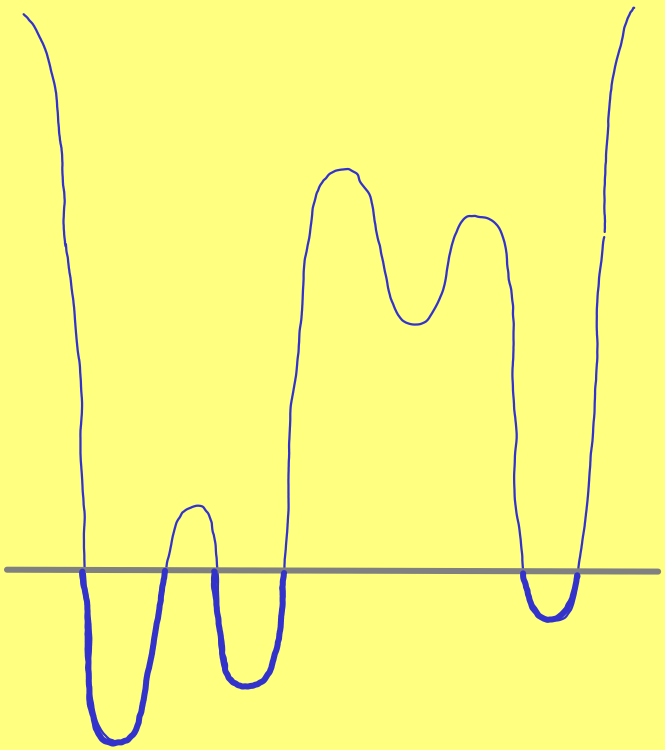
III Contour Stability

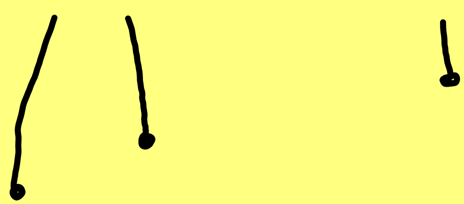
I.1 One-D Functions

function

$f : \mathbb{S}^1 \to \mathbb{R}$

I.1 One-D Functions





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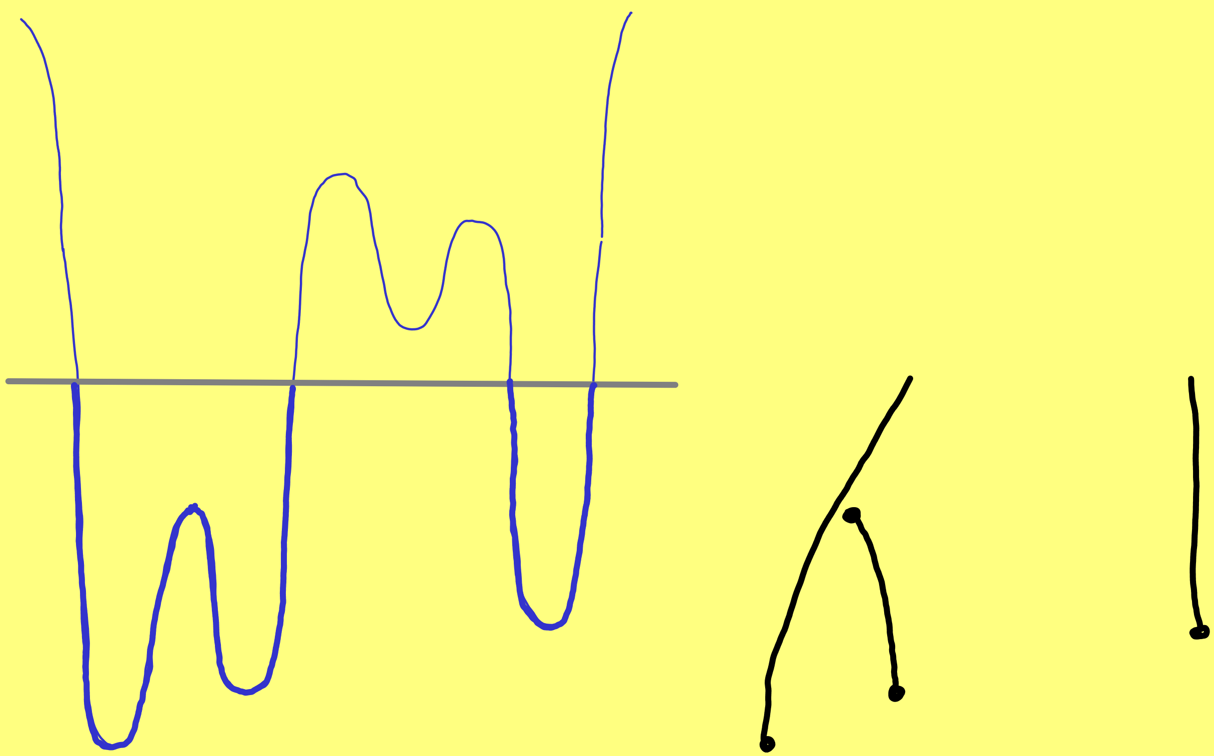

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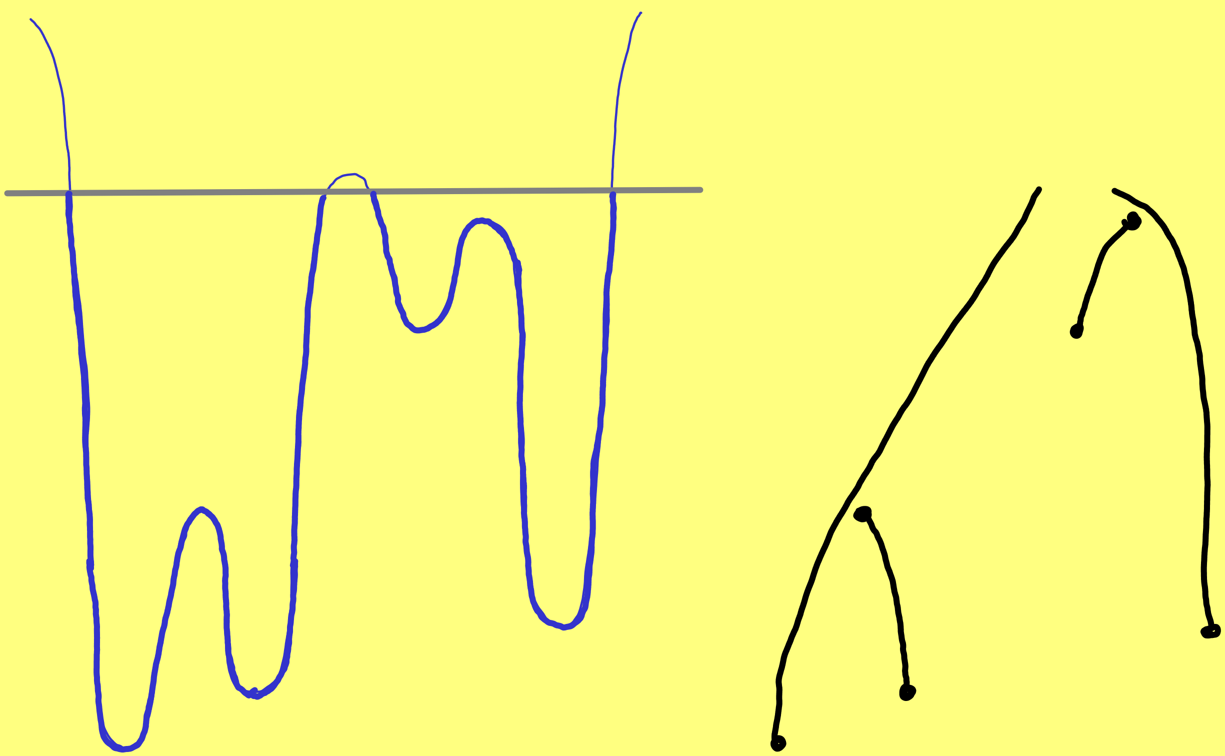

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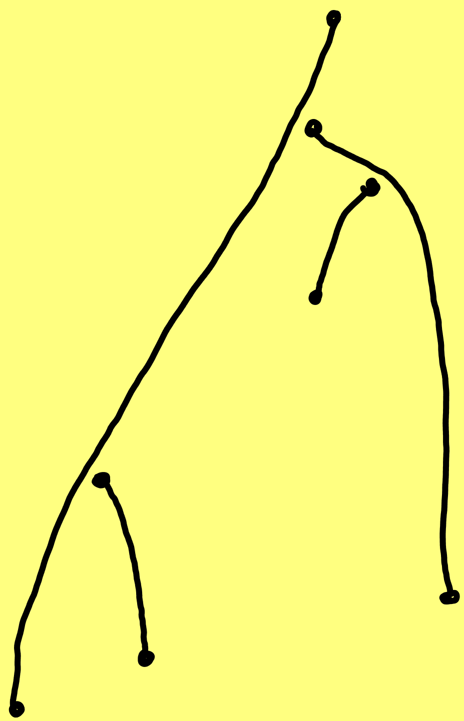
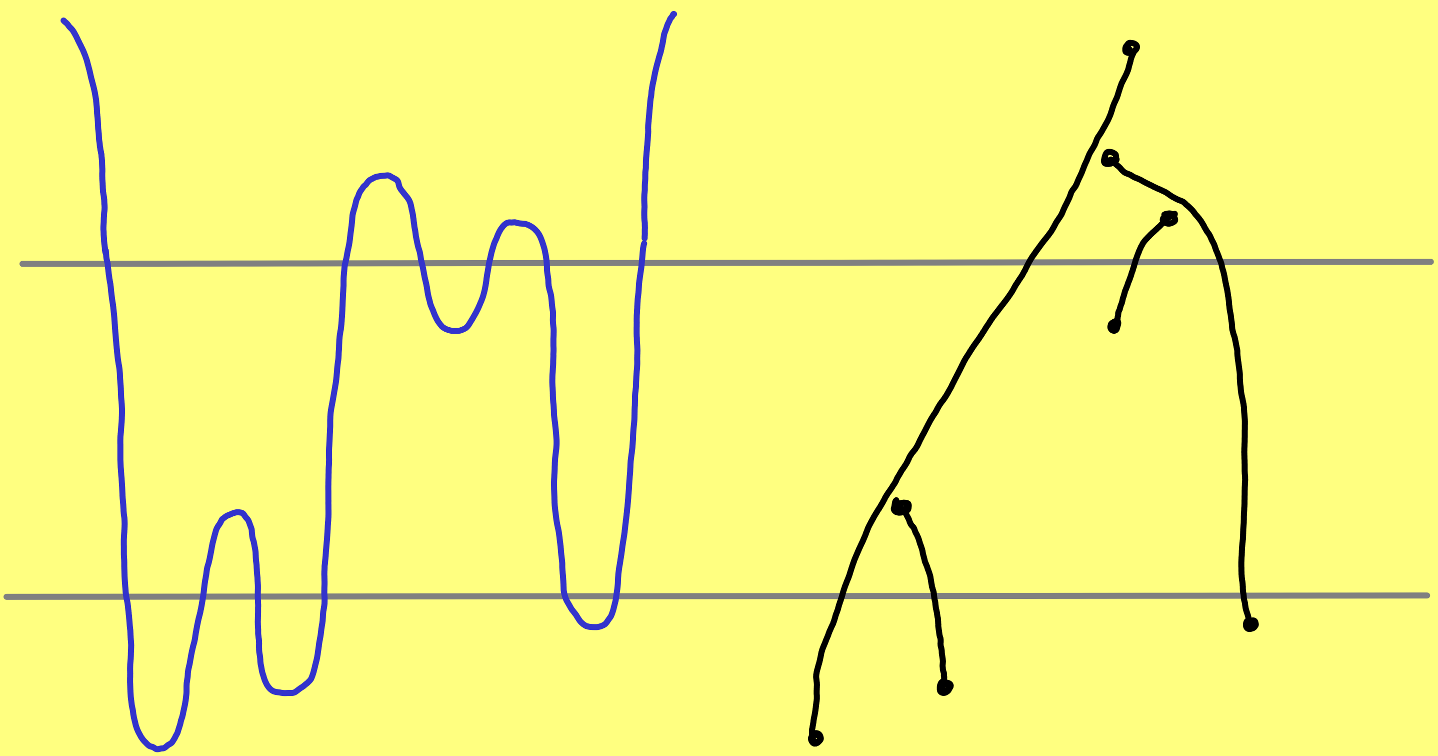


merge tree

I.1 One-D Functions



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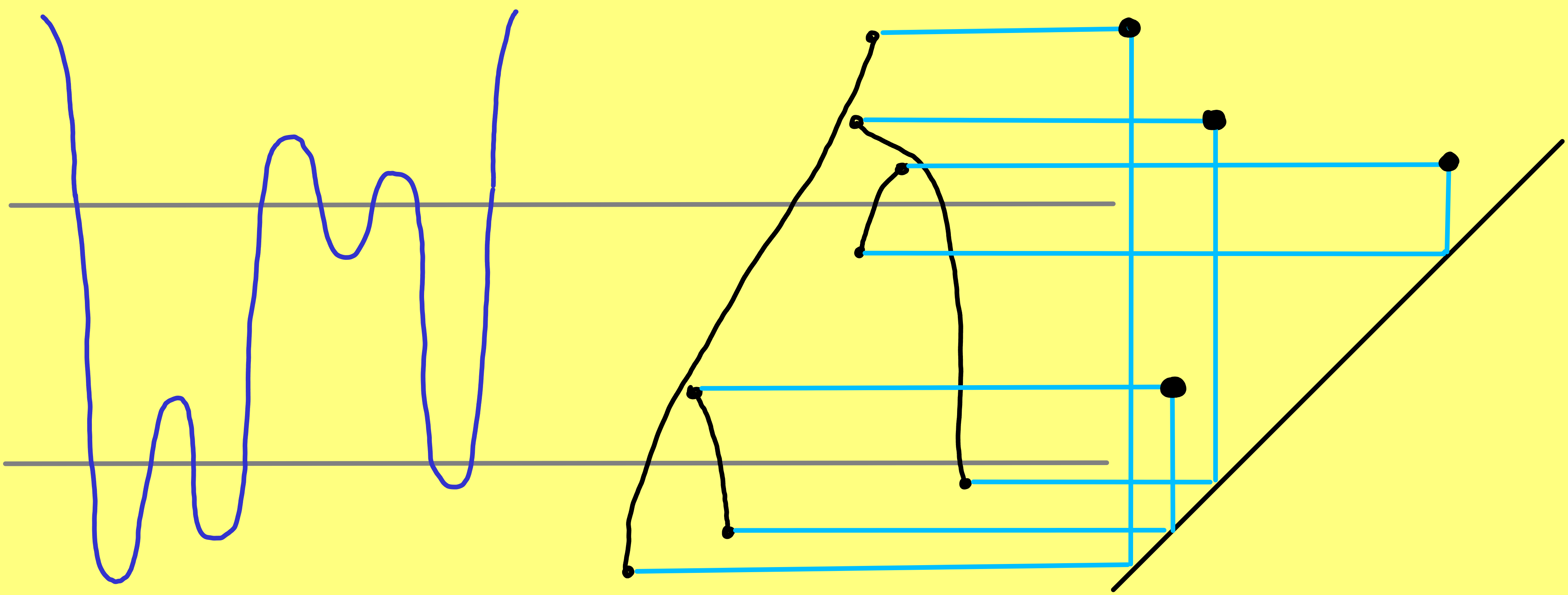

function $f : \mathbb{S}^1 \to \mathbb{R}$ | merge tree | persistence diagram $\mathrm{Dgm}_0(f)$

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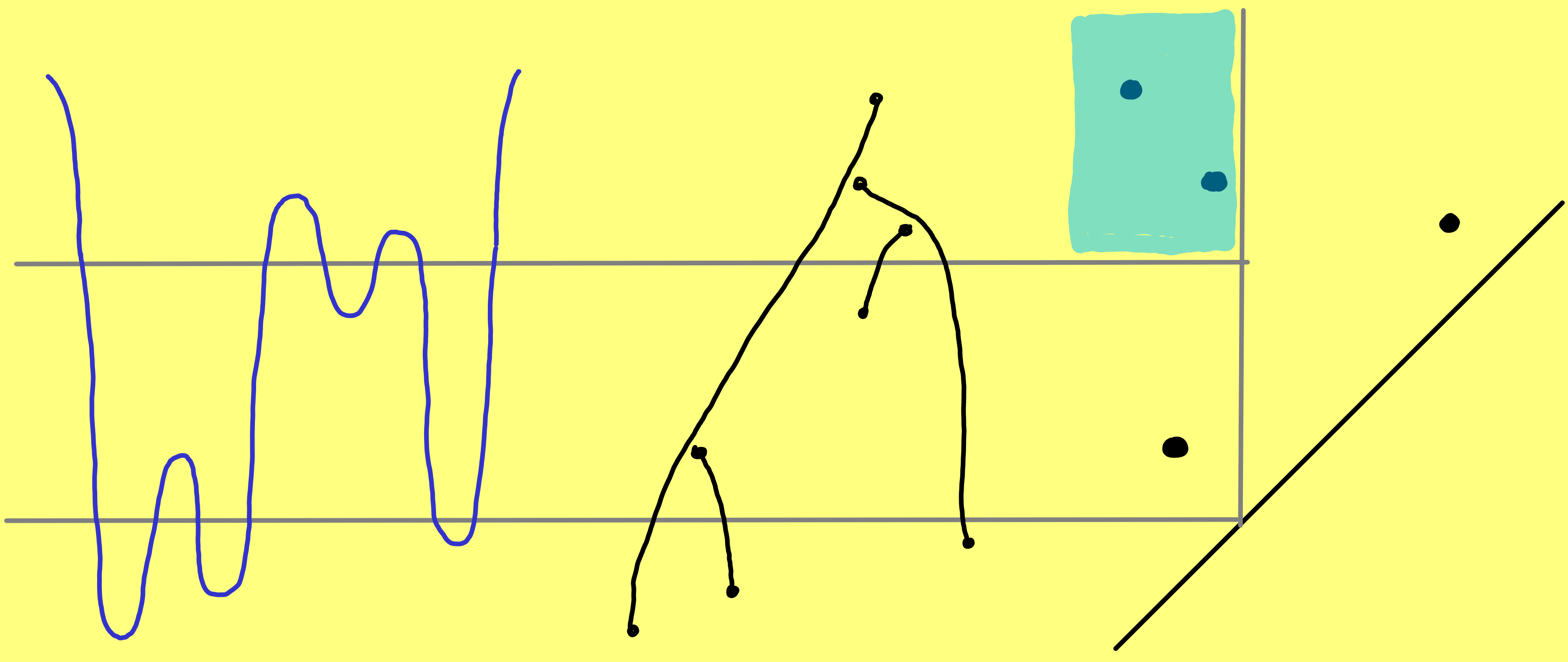

function $\quad$ merge tree $\quad$ persistence diagram

$f : \mathbb{S}^1 \to \mathbb{R}$ $\quad$ $\mathrm{Dgm}_0(f)$

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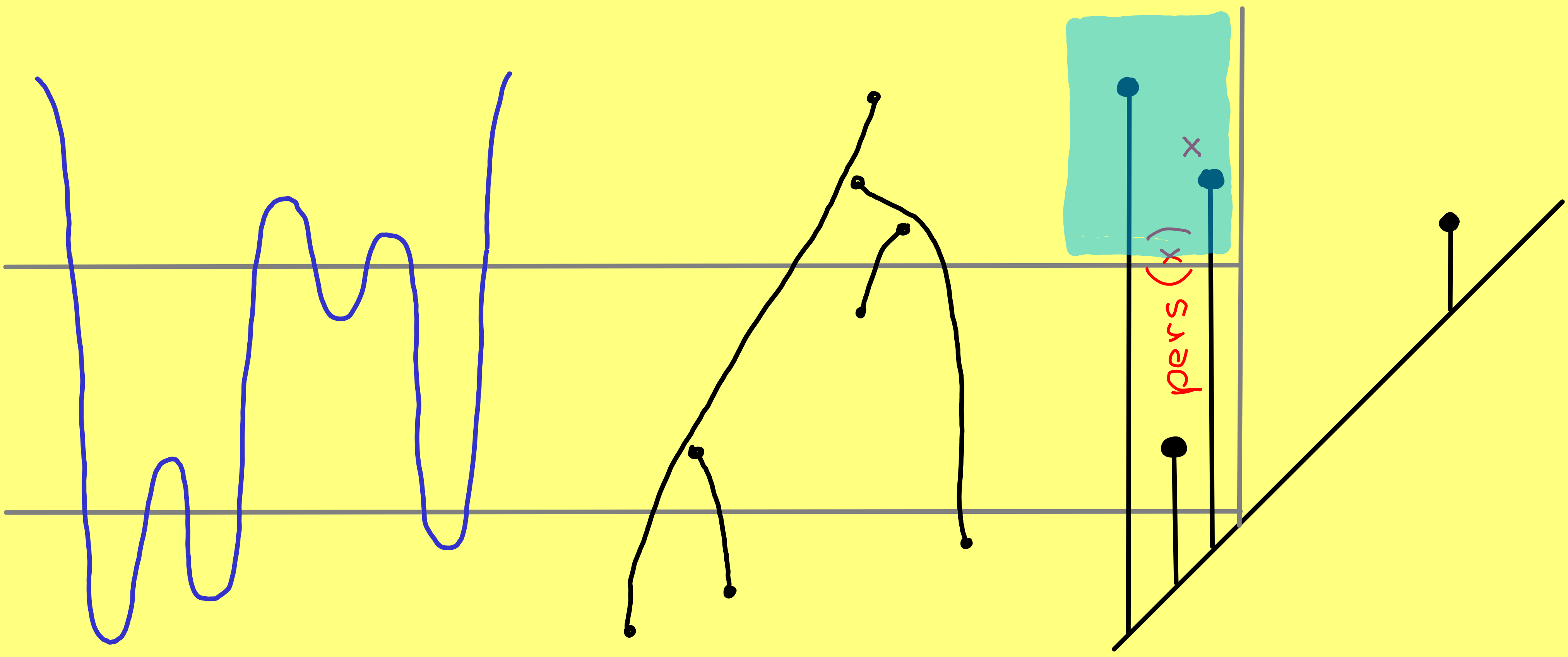

function $f : \mathbb{S}^1 \to \mathbb{R}$

merge tree

persistence diagram $Dgm_0(f)$

I.2 Filtrations

$$\ldots \subseteq K_{i-1} \subseteq K_i \subseteq \ldots \subseteq K_{j-1} \subseteq K_j \subseteq \ldots$$

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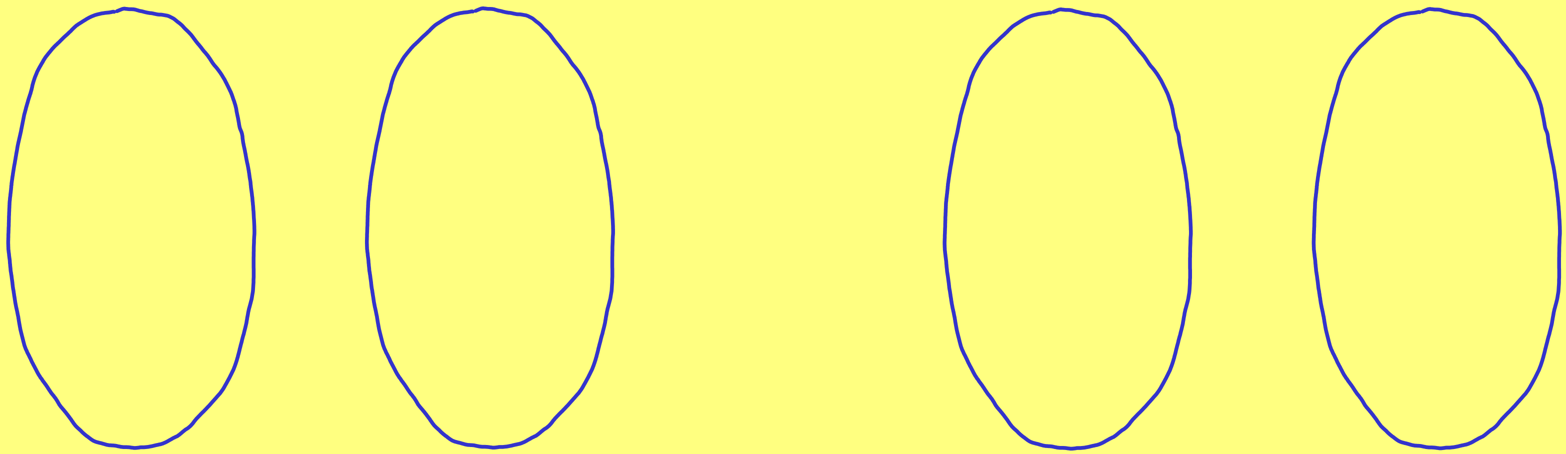

$$\ldots \to H_p(K_{i-1}) \to H_p(K_i) \to \ldots \to H_p(K_{j-1}) \to H_p(K_j) \to \ldots$$

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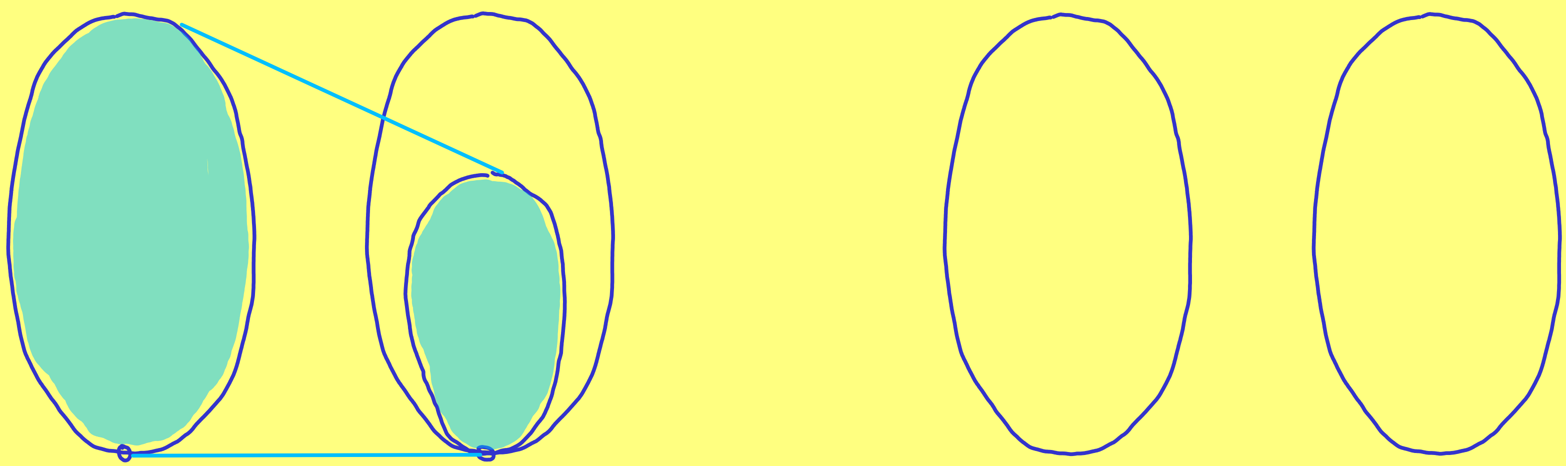

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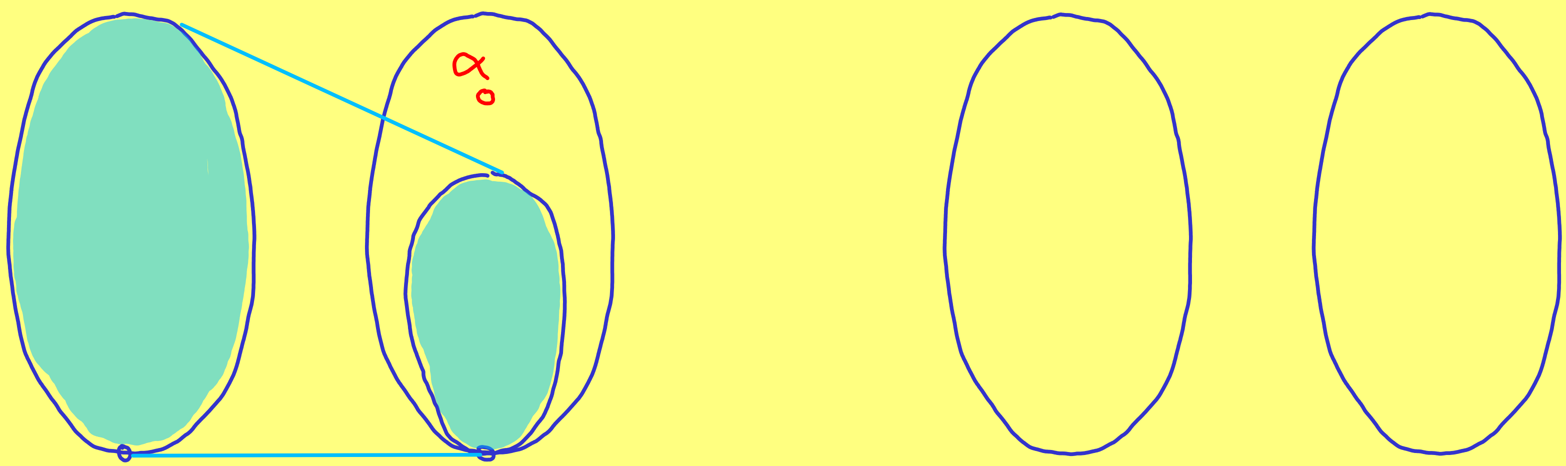

$$\ldots \longrightarrow H_p(K_{i-1}) \longrightarrow H_p(K_i) \longrightarrow \ldots \longrightarrow H_p(K_{j-1}) \longrightarrow H_p(K_j) \longrightarrow \ldots$$

α is born at K_i

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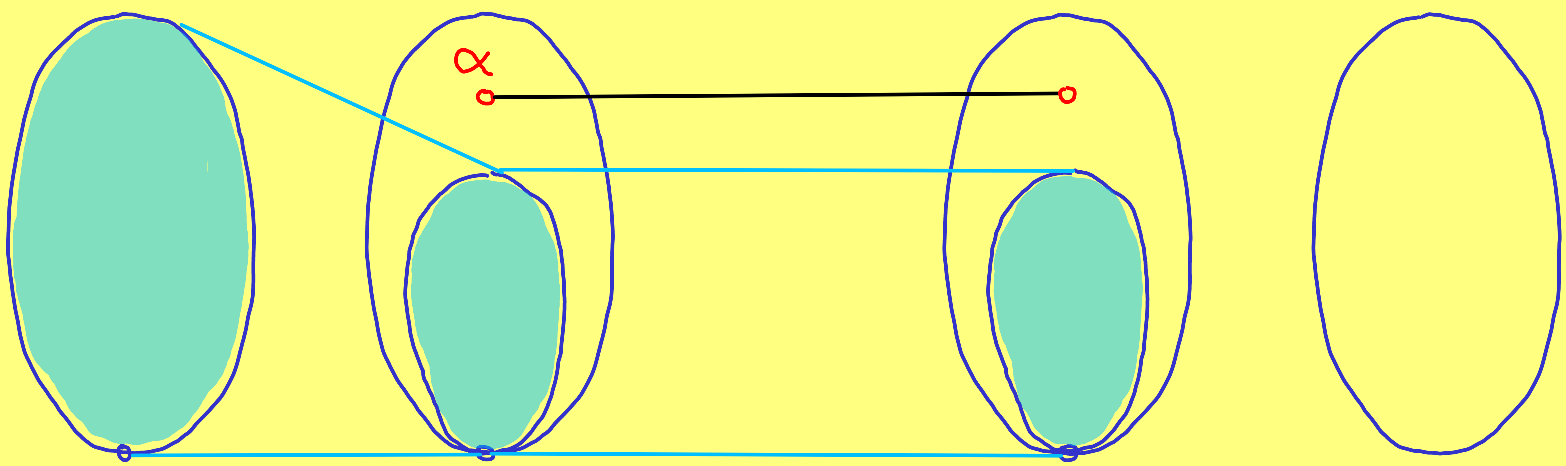

$$\ldots \rightarrow H_p(K_{i-1}) \rightarrow H_p(K_i) \rightarrow \ldots \rightarrow H_p(K_{j-1}) \rightarrow H_p(K_j) \rightarrow \ldots$$

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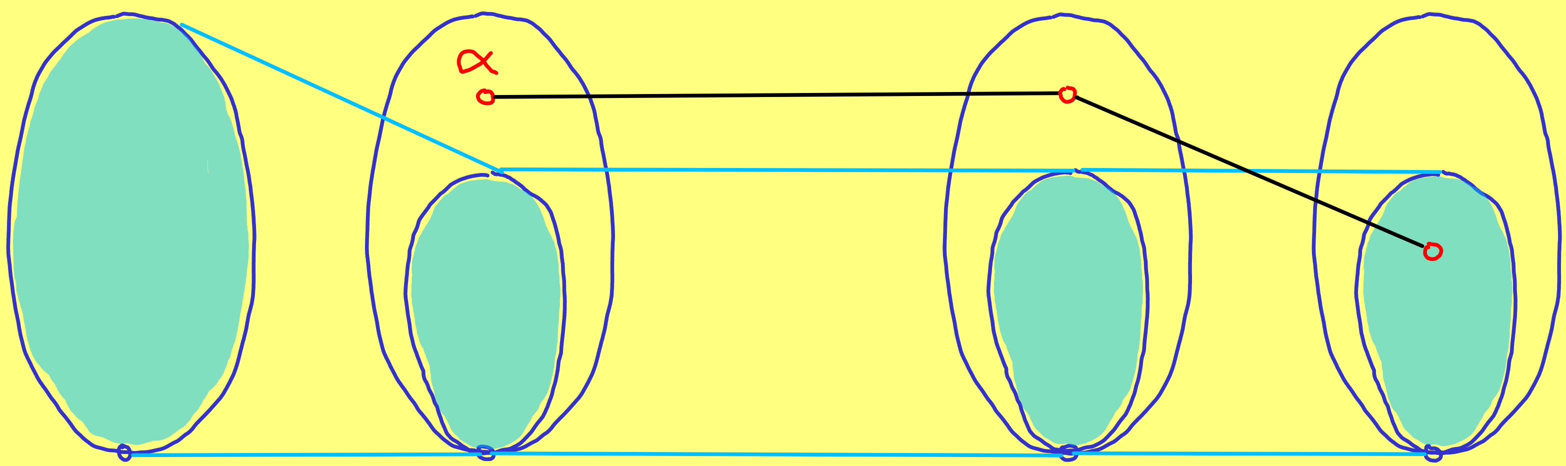
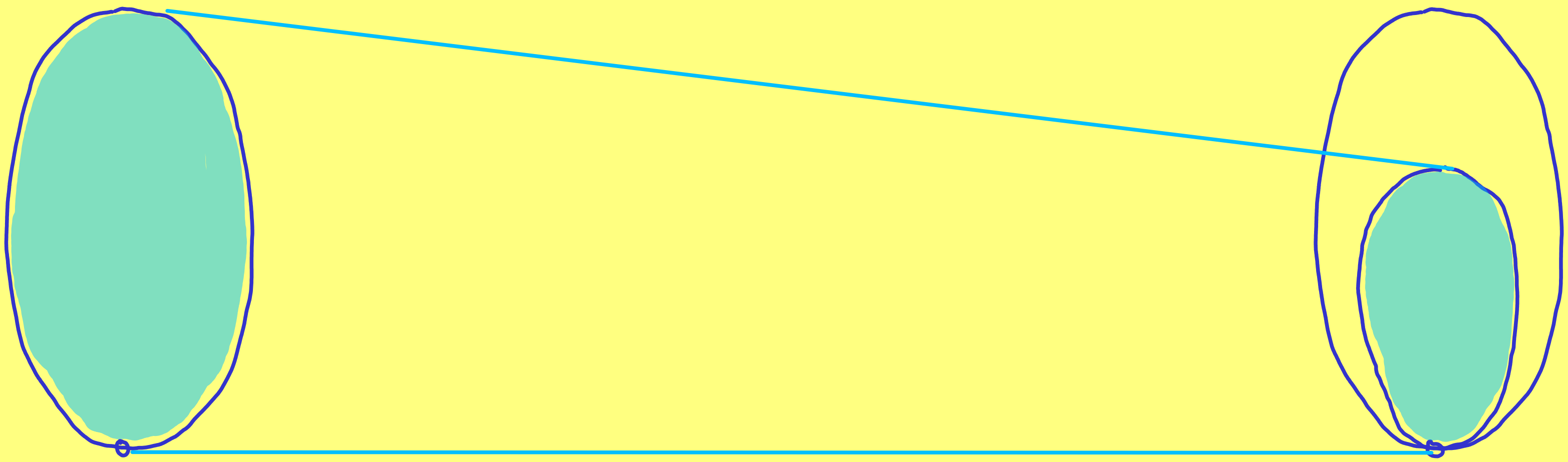


$$\dots \to H_p(K_{i-1}) \to H_p(K_i) \to \dots \to H_p(K_{j-1}) \to H_p(K_j) \to \dots$$

α is born at K_i and dies entering K_j

I.2 FILTRATIONS

$$\ldots \subseteq K_{i-1} \subseteq K_i \subseteq \ldots \subseteq K_{j-1} \subseteq K_j \subseteq \ldots$$



$$H_p(K_{i-1}) \to \quad \ldots \quad \to H_p(K_j)$$

I.2 Filtrations

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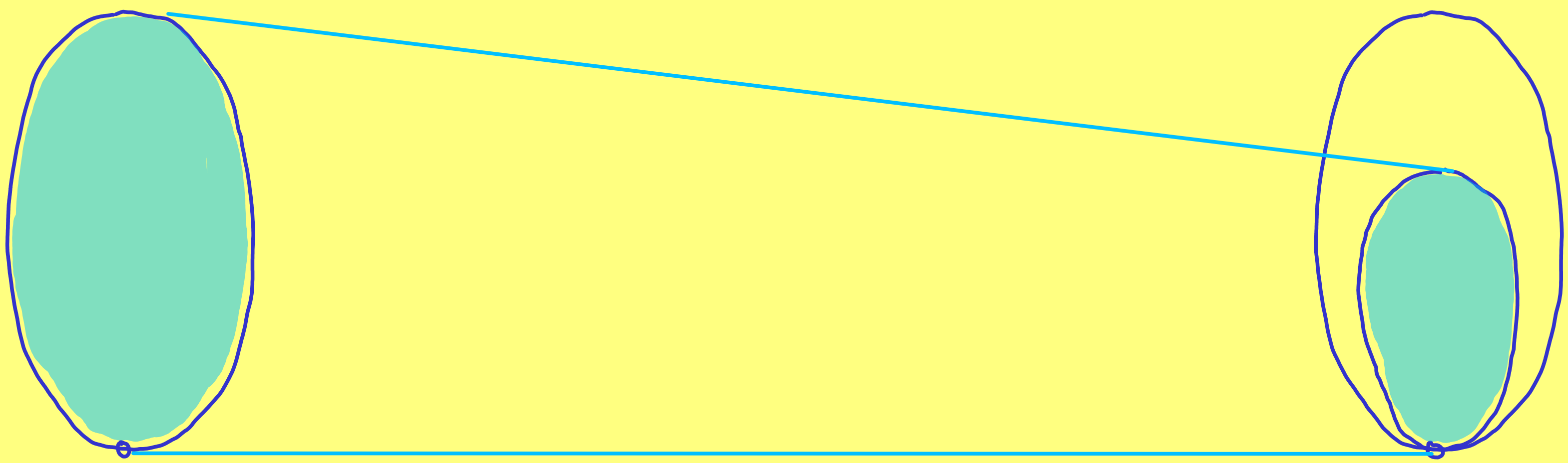

$$h_p^{i-1,j} : H_p(K_{i-1}) \rightarrow \quad \ldots \quad \rightarrow H_p(K_j)$$

induced homomorphism

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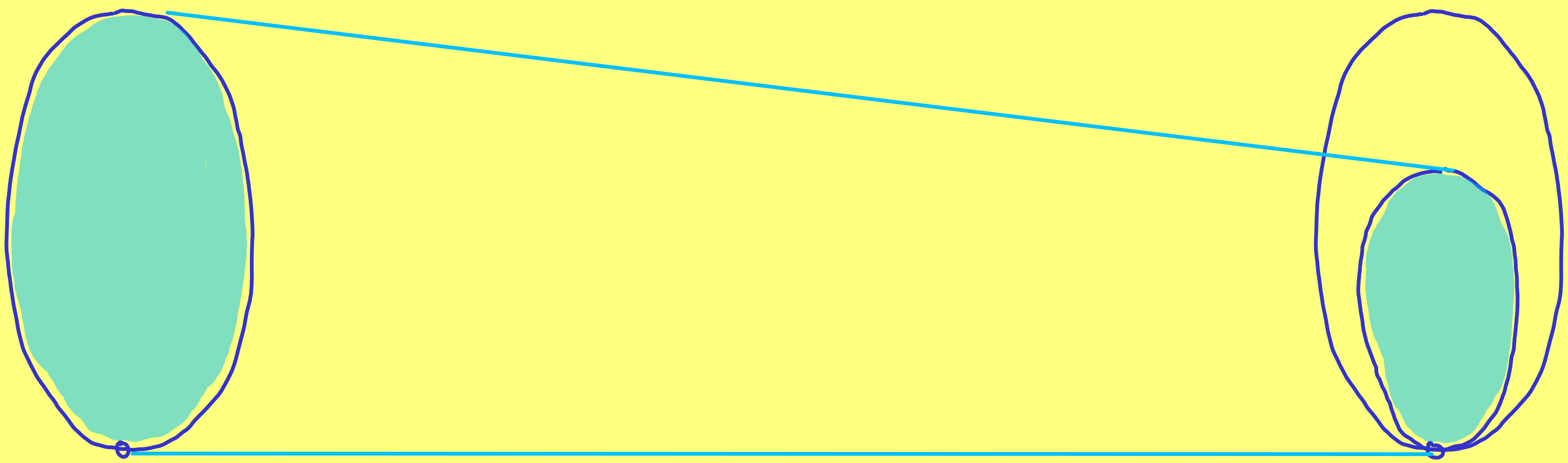
$$\ldots \subseteq K_{i-1} \subseteq K_i \subseteq \ldots \subseteq K_{j-1} \subseteq K_j \subseteq \ldots$$



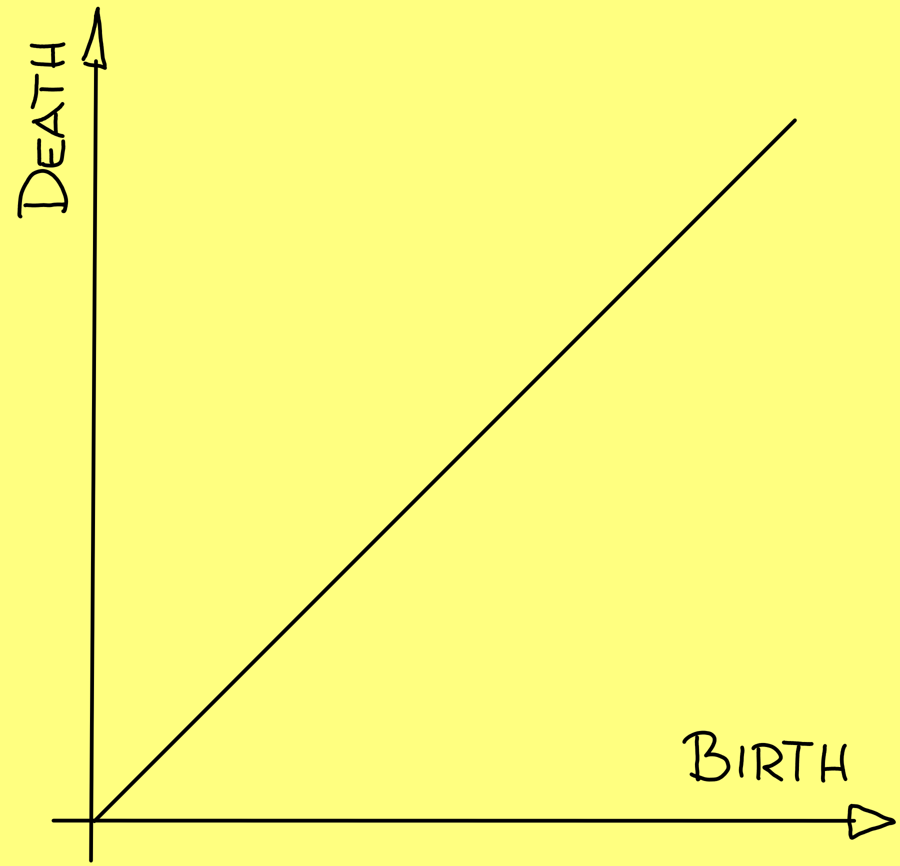
$$h_p^{i-1,j} : H_p(K_{i-1}) \to \quad \ldots \quad \to H_p(K_j)$$

induced homomorphism

$\operatorname{im} h_p^{i-1,j}$ is persistent homology group

I.3 Diagrams

$\mathrm{Dgm}_p(f)$:



I.3 DIAGRAMS

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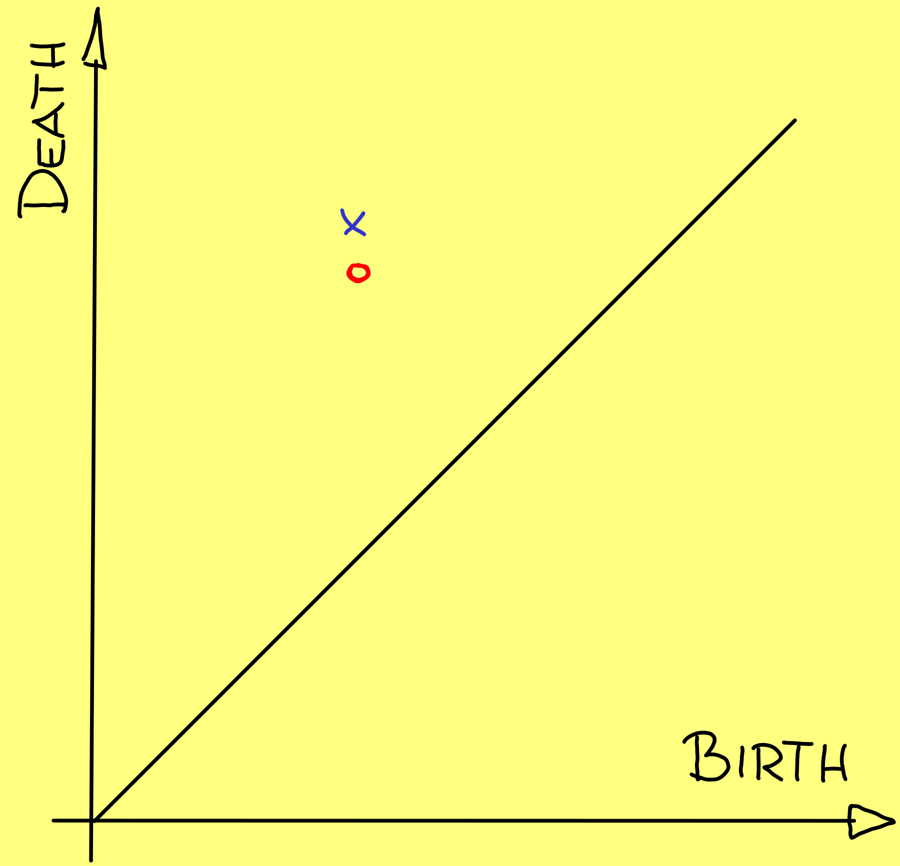

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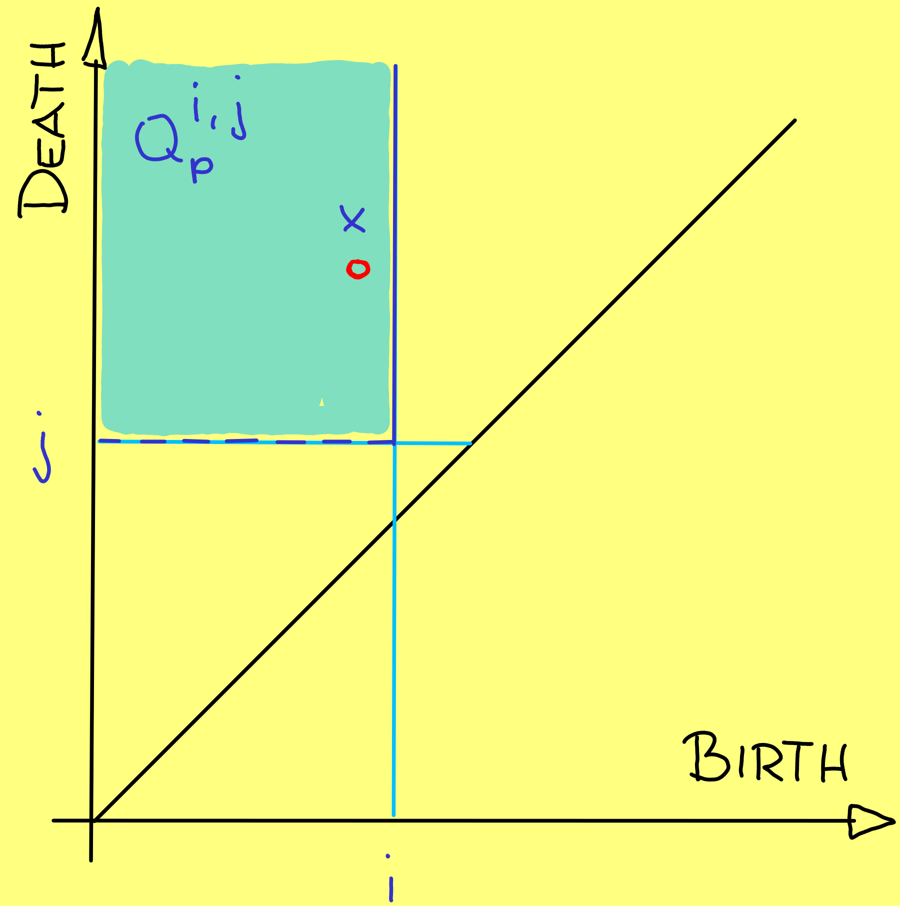

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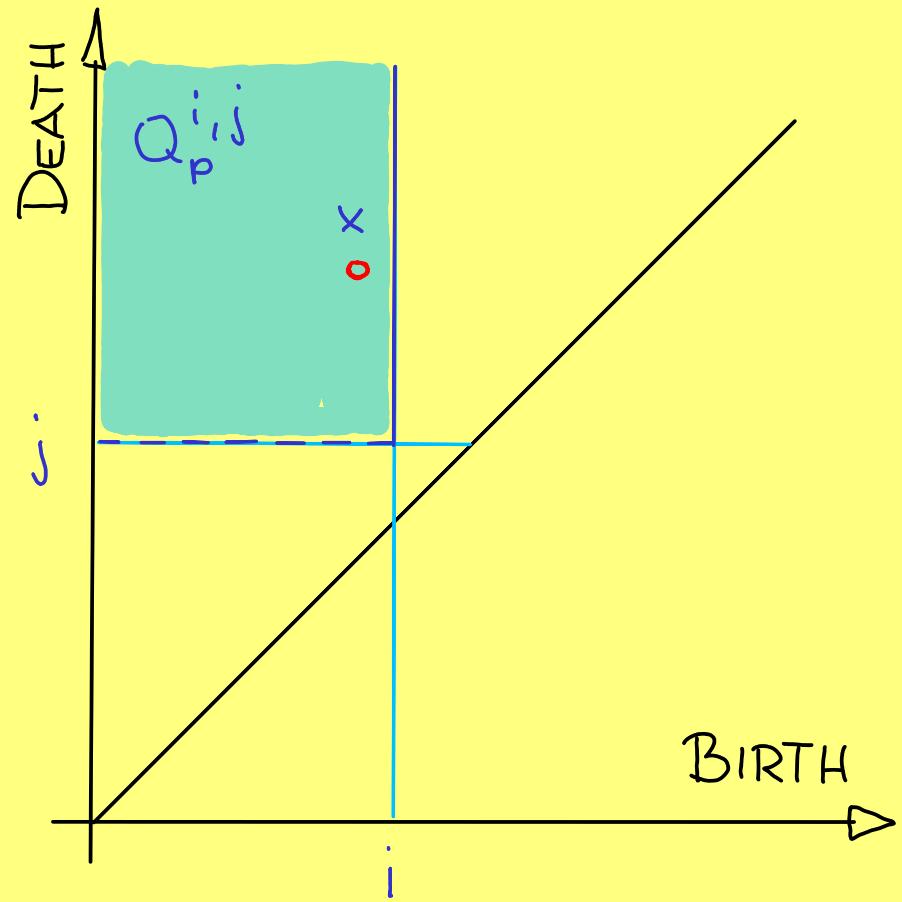
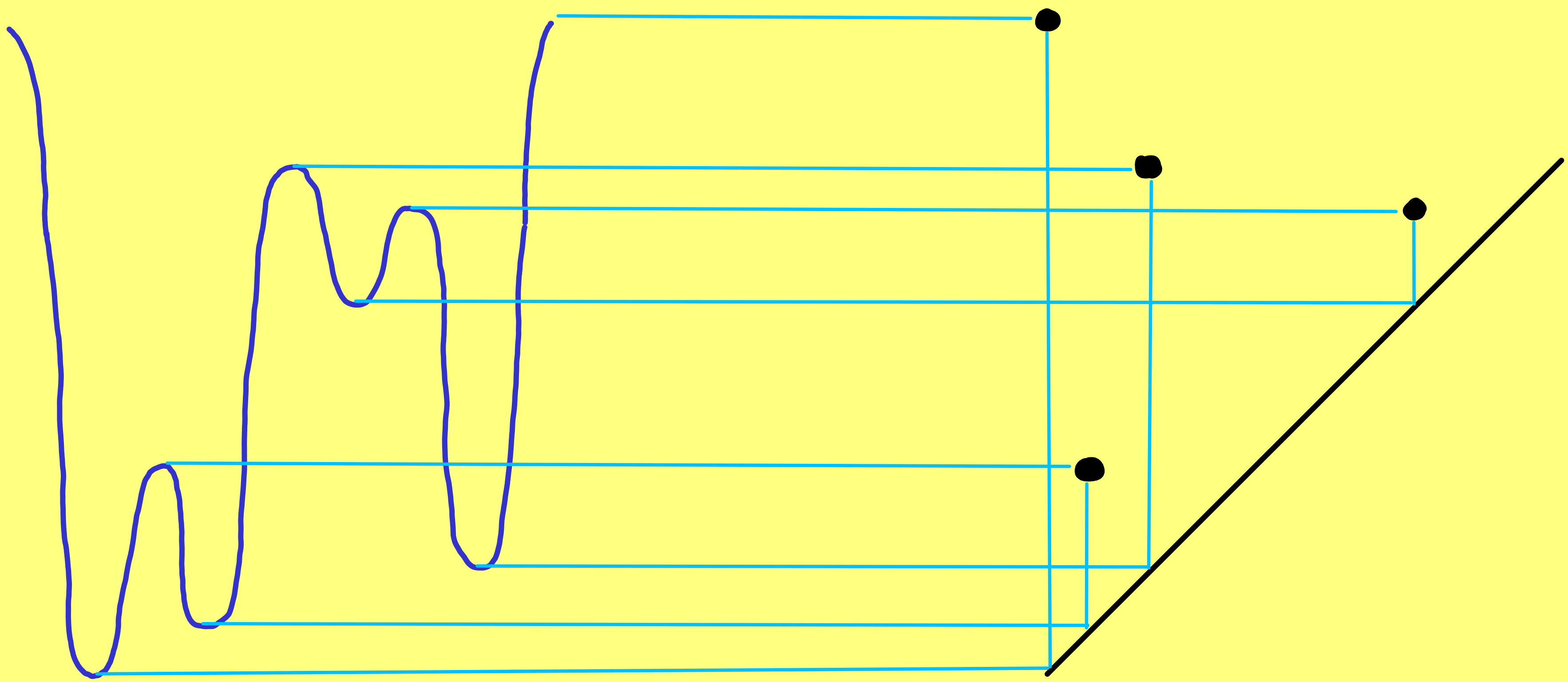
$\mathrm{Dgm}_p(f)$:



FUNDAMENTAL LEMMA OF PERSISTENT HOMOLOGY.

$\#\text{points in } Q_p^{i,j} = \operatorname{rank} \operatorname{im} h_p^{i,j}$.

I.4 Bottleneck Distance



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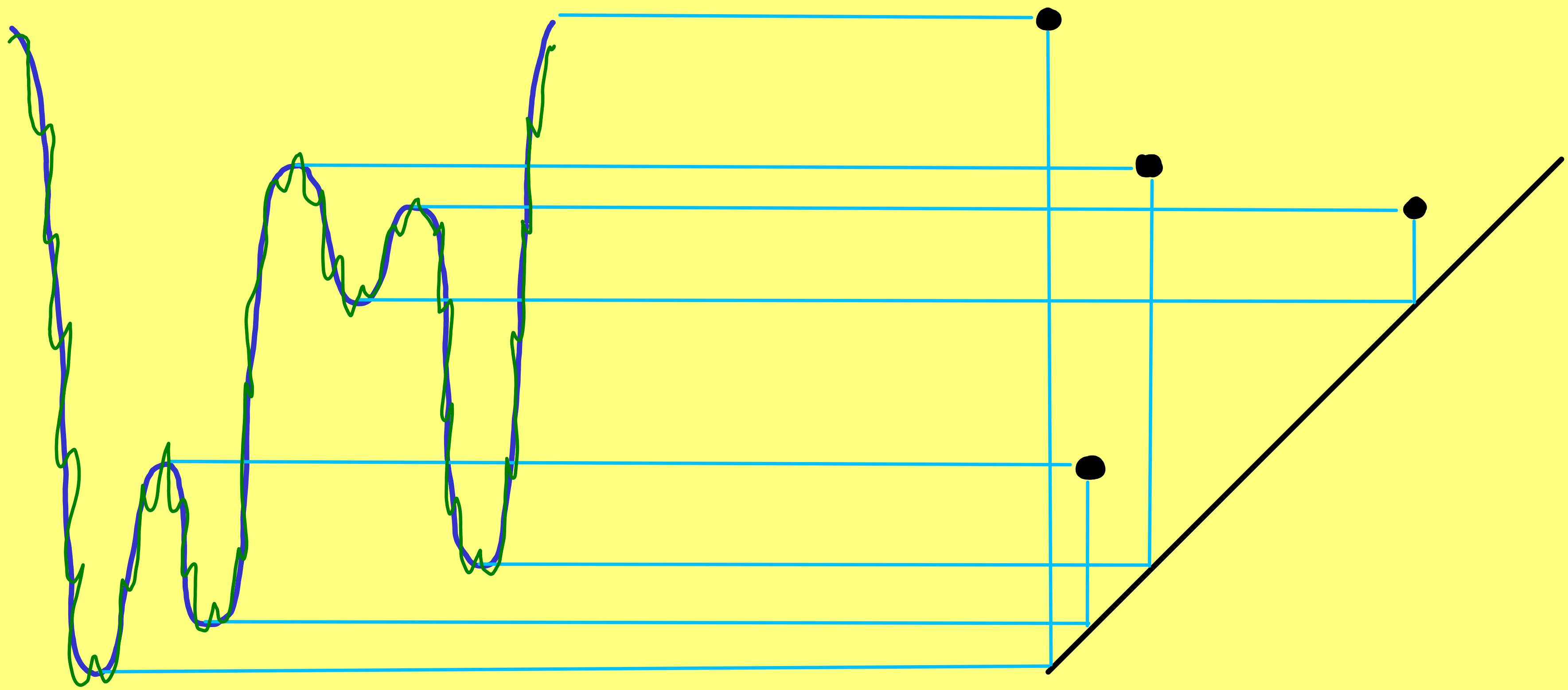

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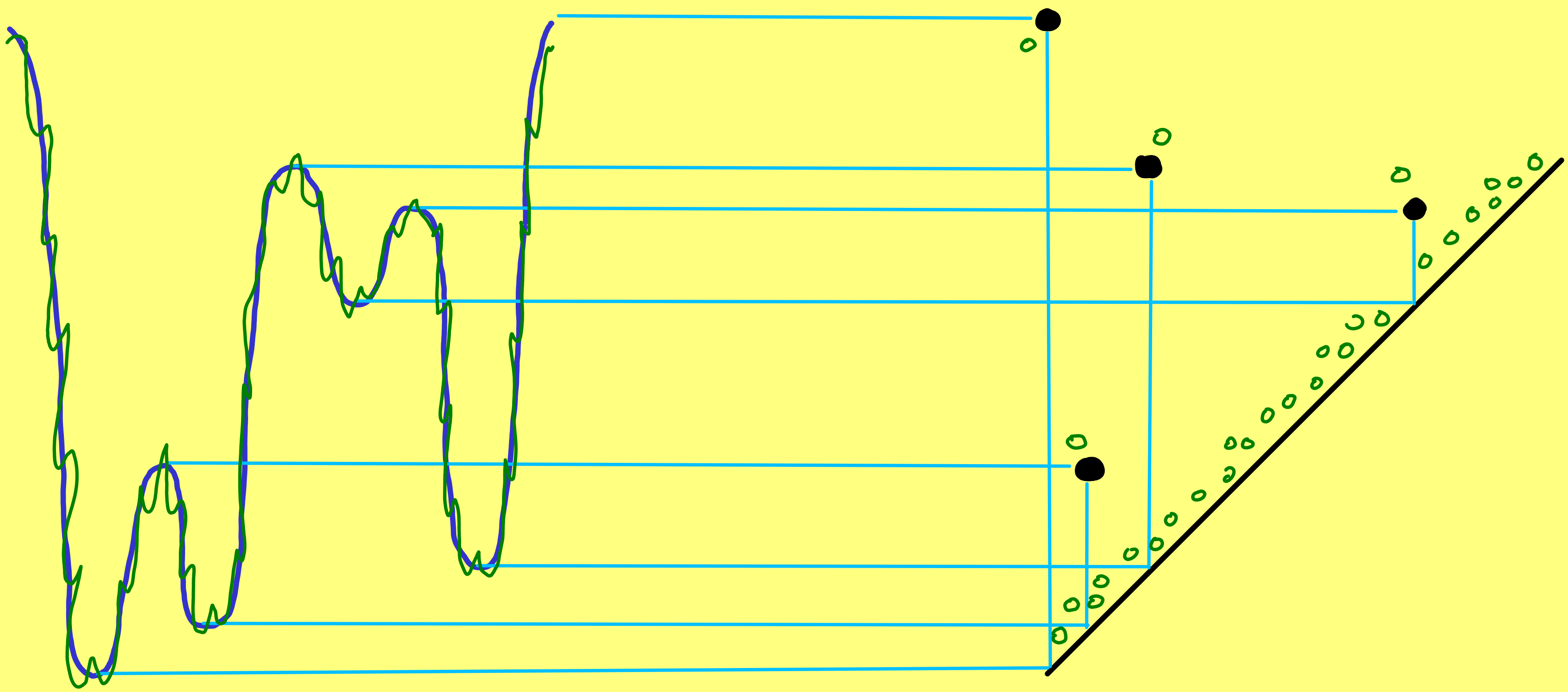

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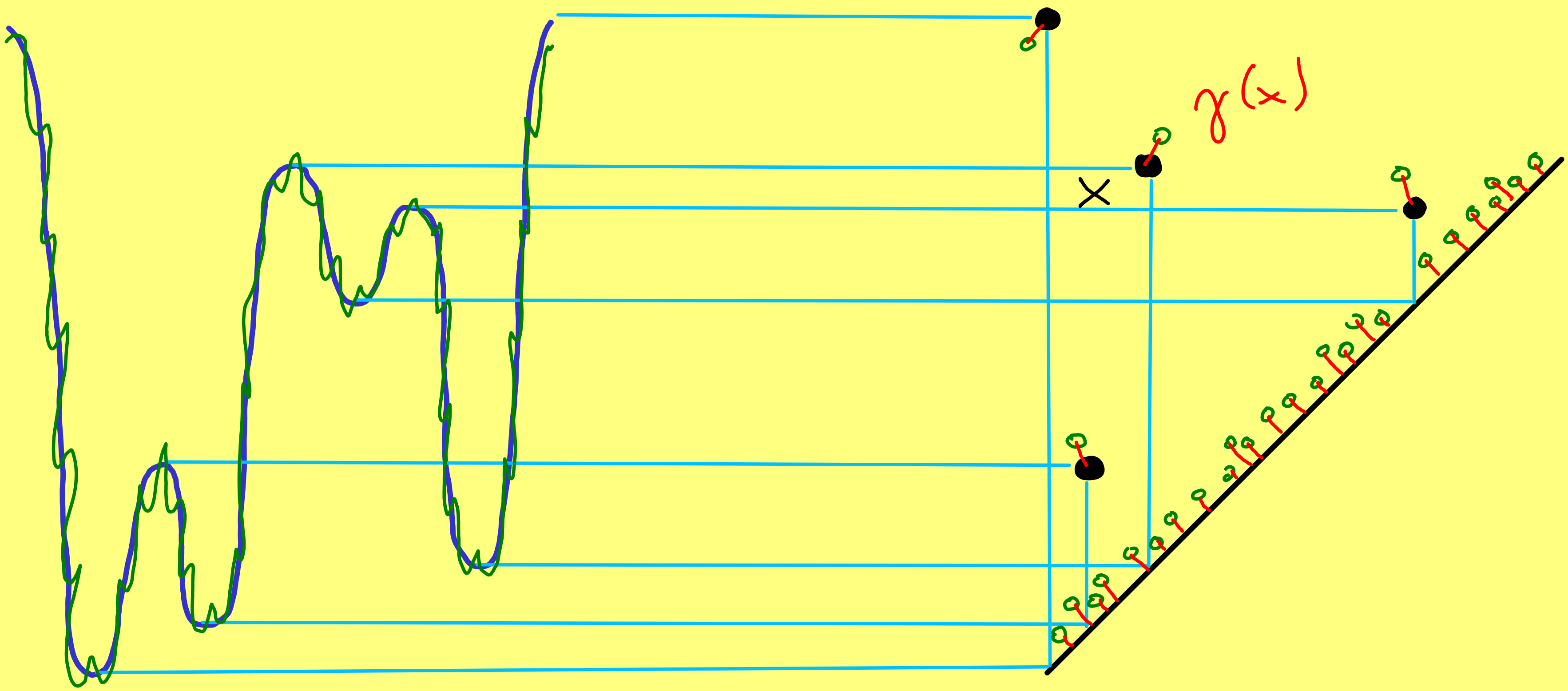


The bottleneck distance between the diagrams is

$$W_\infty(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) = \inf_\gamma \sup_x \|x - \gamma(x)\|_\infty .$$

I.4 BOTTLENECK DISTANCE

L_∞-STAB. THM. For tame functions $f, g : \mathbb{X} \to \mathbb{R}$ the bottleneck distance between their diagrams is bounded by the max-difference between the functions,

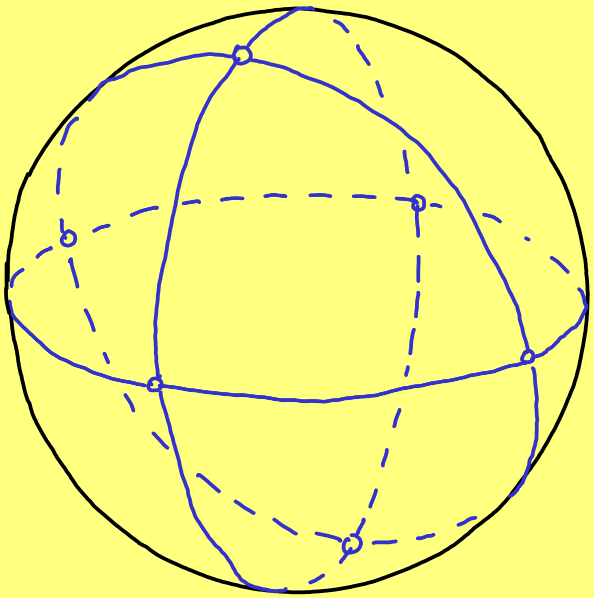
$$W_\infty(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) \leq \|f - g\|_\infty .$$

I Persistent Homology

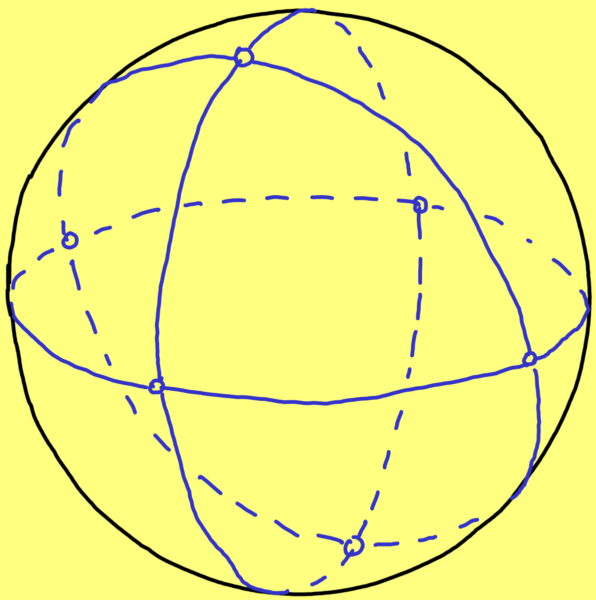
II L_p-stability and Somites

III Contour Stability

II.1 Size and Growth



II.1 Size and Growth



mesh (K) = max. diameter of a simplex

size (K) = number of simplices

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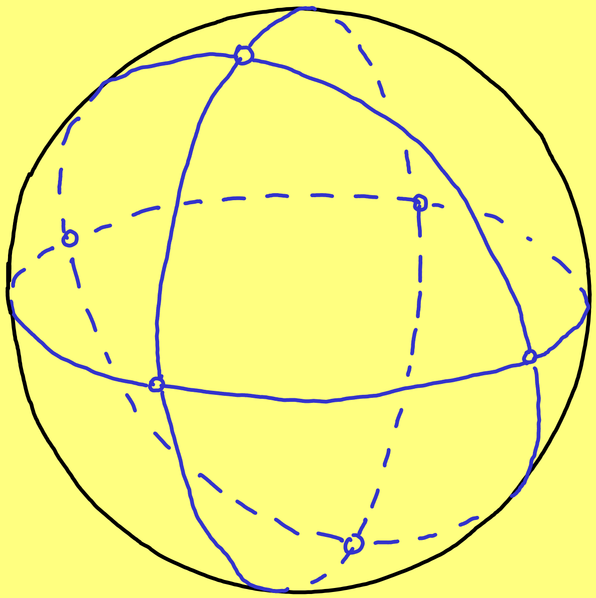

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$$N(r) = \min_{\text{mesh}(K) \le r} \text{size}(K)$$

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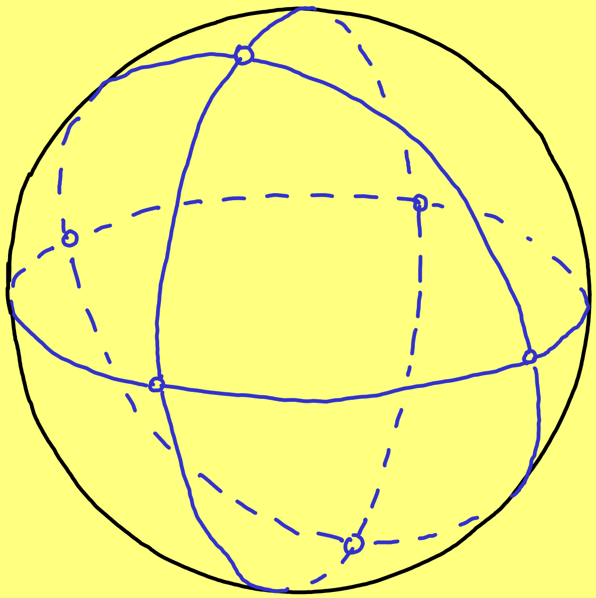

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$$N(r) = \min_{\text{mesh}(K) \leq r} \text{size}(K)$$

The triangulation grows polynomially if $\exists$ constants C, M s.t.

$$N(r) \leq C/r^M.$$

II.2 Total Persistence

The q-th total persistence of f is

$$\mathrm{Pers}_q(f) = \sum_p \sum_x \mathrm{pers}(x)^q$$

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f Lipschitz and $N(r) \leq C / r^M$

$\Longrightarrow$ $\mathrm{Pers}_q(f) \leq \mathrm{const}$ for all $q > M$

II.2 Total Persistence

TP-STAB. THM. Let $\mathbb{X}$ be a compact metric space with polynomially growing triangulation and $f, g : \mathbb{X} \to \mathbb{R}$ Lipschitz functions. Then

$$\mathrm{Pers}_q(f) - \mathrm{Pers}_q(g) \leq \mathrm{const} \cdot \|f - g\|_\infty$$

for every $q > M + 1$.

II.2 SIMPLIFICATION

normalized to $\operatorname{amp}(f) = \max_x f(x) - \min_x f(x) = 1$

II.2 SIMPLIFICATION

An ε-simplification of f is a function $f_\varepsilon : \mathbb{X} \to \mathbb{R}$ with $\| f - f_\varepsilon \|_\infty \leq \varepsilon$ whose diagrams $\mathrm{Dgm}_p(f_\varepsilon)$ are same as $\mathrm{Dgm}_p(f)$ without points of persistence at most ε.

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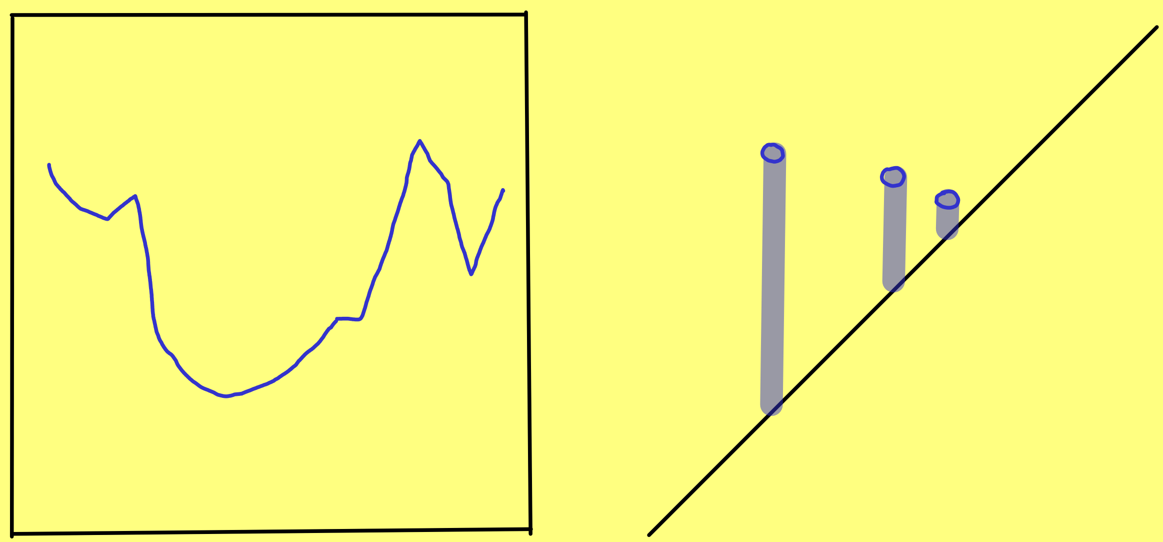

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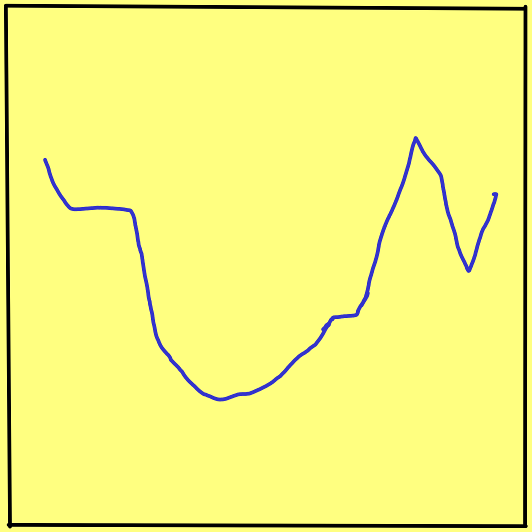

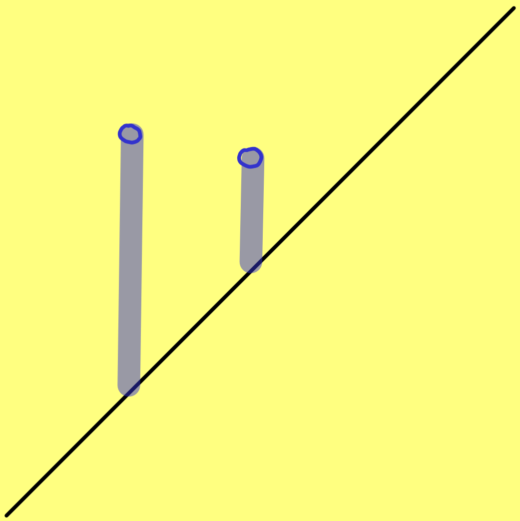

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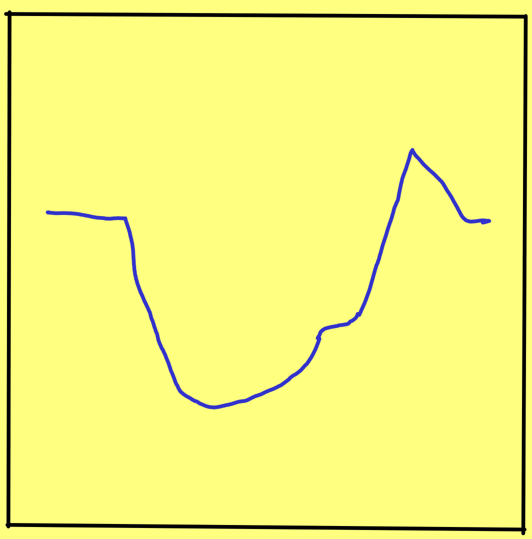

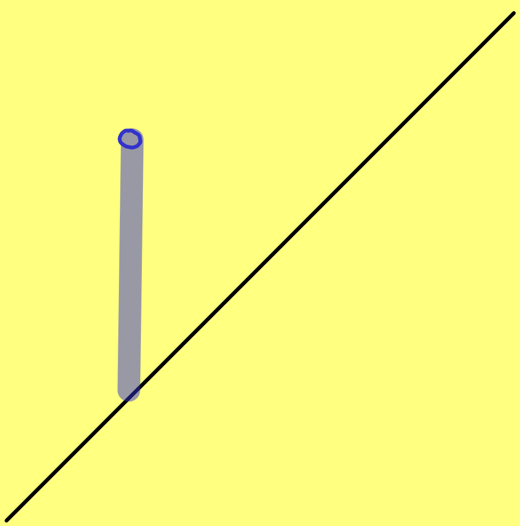

II.2 Simplification

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$$M_1(f) = \int_{\varepsilon=0}^{1} M_0(f_\varepsilon)\, d\varepsilon$$

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$$M_1(f) = \int_{\varepsilon=0}^{1} M_0(f_\varepsilon)\, d\varepsilon$$

$$M_2(f) = \int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\, d\varepsilon$$

etc.

II.2 SIMPLIFICATION

$f : \mathbb{S}^1 \to \mathbb{R}$

$$M_0(f) = \frac{1}{2} \,\#\text{crit. pts.} = \mathrm{Pers}_0(f)$$

$$M_1(f) = \int_{\varepsilon=0}^{1} M_0(f_\varepsilon)\, d\varepsilon = \mathrm{Pers}_1(f)$$

$$M_2(f) = \int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\, d\varepsilon = \mathrm{Pers}_2(f)$$

etc.

II.2 Simplification

$$f : \mathbb{S}^1 \to \mathbb{R}$$

not stable

$$M_0(f) = \frac{1}{2}\,\#\text{crit. pts.} = \mathrm{Pers}_0(f)$$

$$M_1(f) = \int_{\varepsilon=0}^{1} M_0(f_\varepsilon)\,d\varepsilon = \mathrm{Pers}_1(f)$$

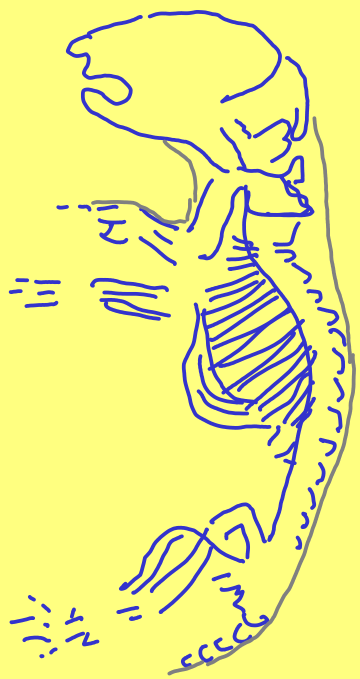
stable

$$M_2(f) = \int_{\varepsilon=0}^{1} M_1(f_\varepsilon)\,d\varepsilon = \mathrm{Pers}_2(f)$$

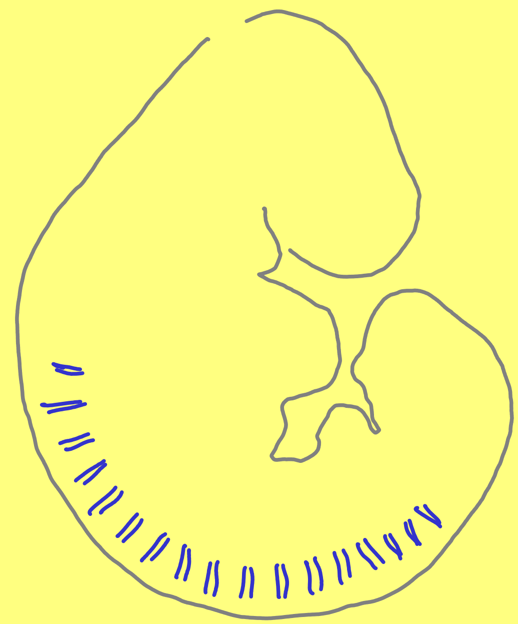
etc.

II.3 Rhythmic Gene Expression

adult mouse

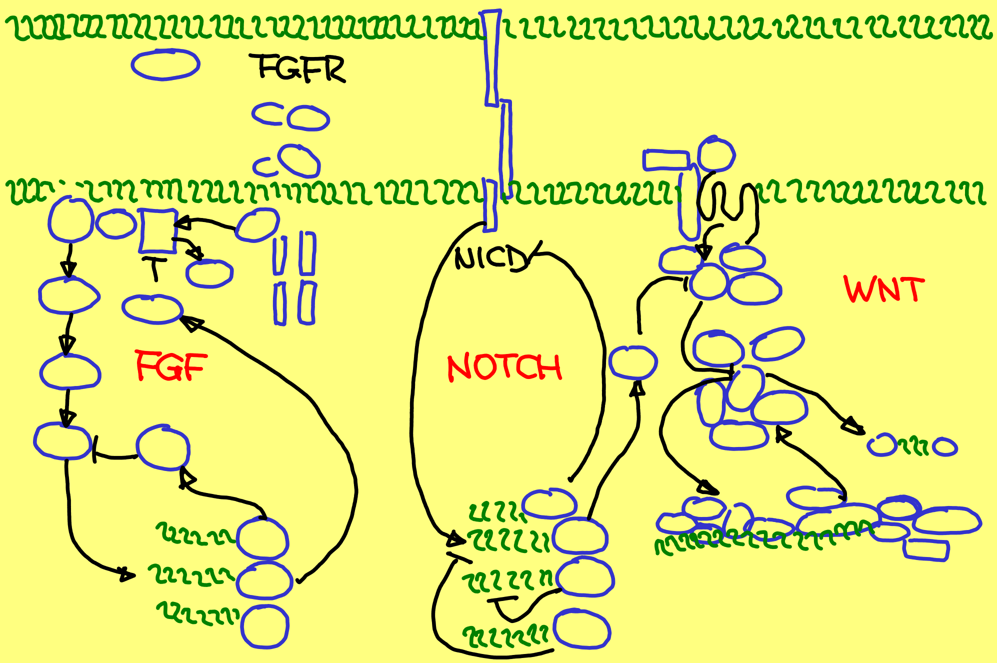


mouse embryo



somite development is a rythmic process

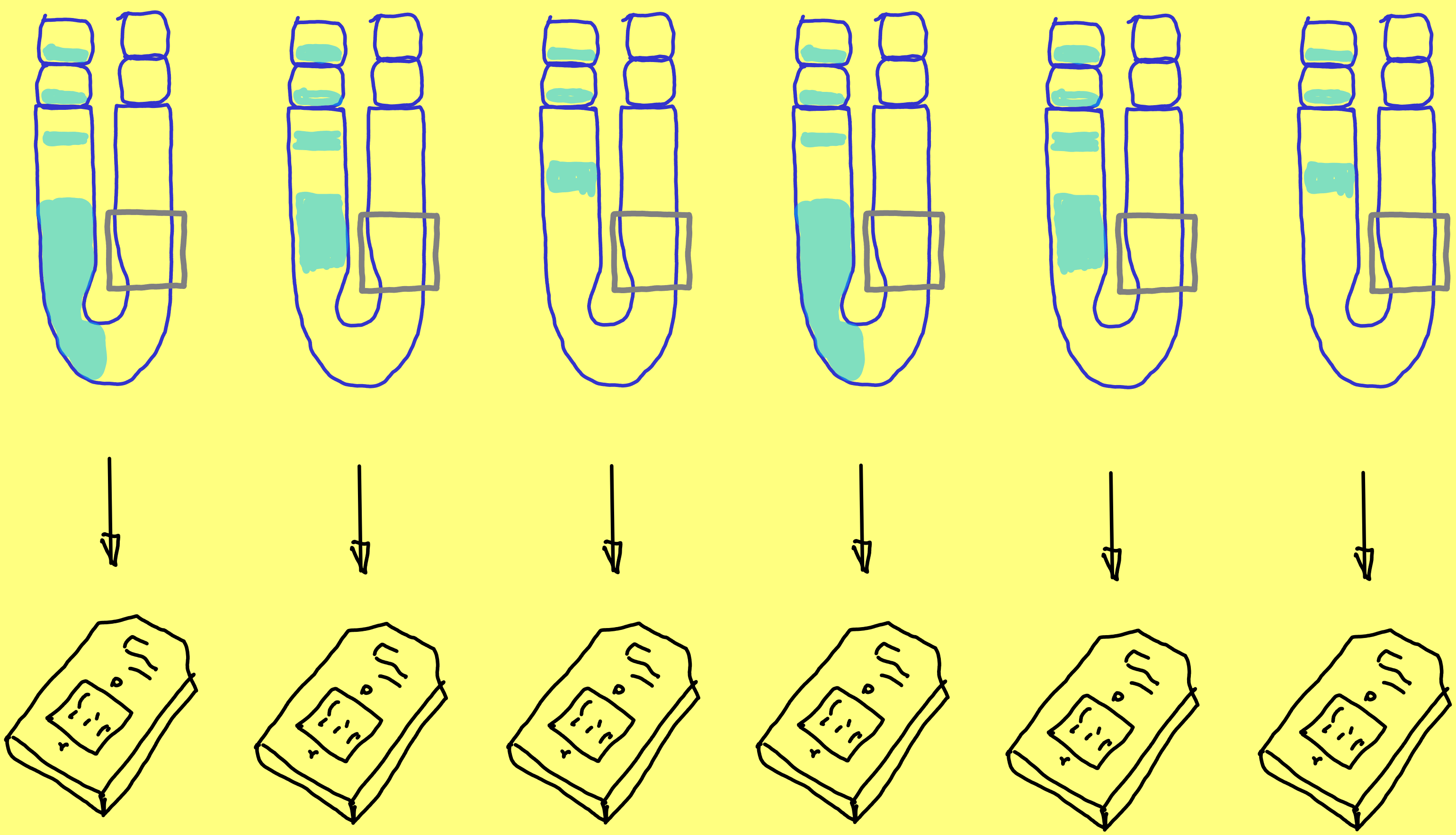
II.3 Rhythmic Gene Expression



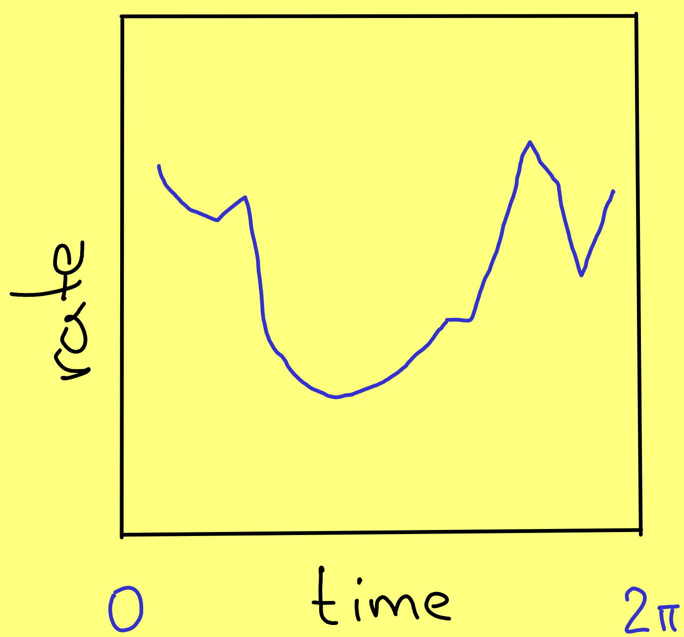
the proposed clock

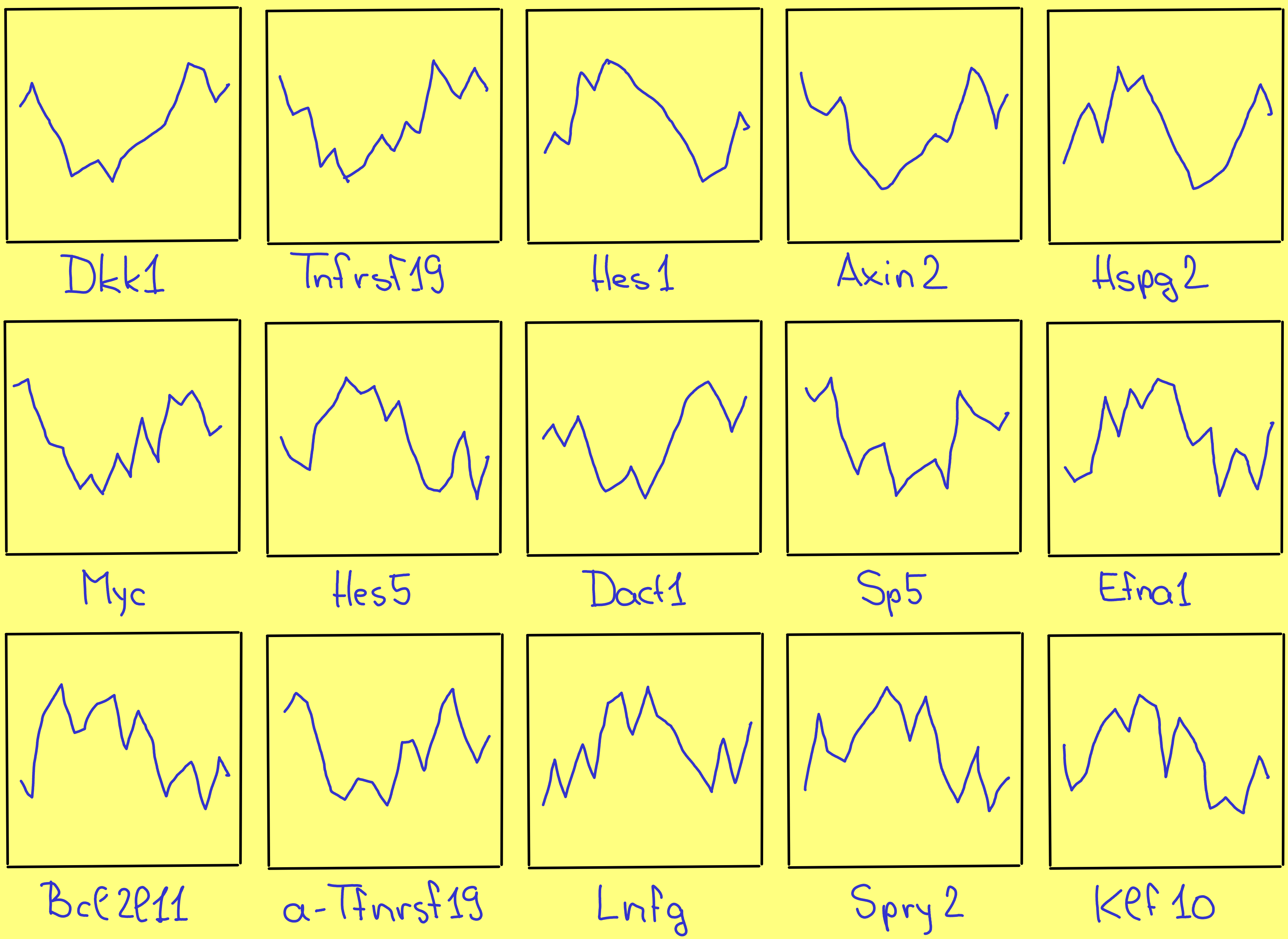
II.3 Rhythmic Gene Expression

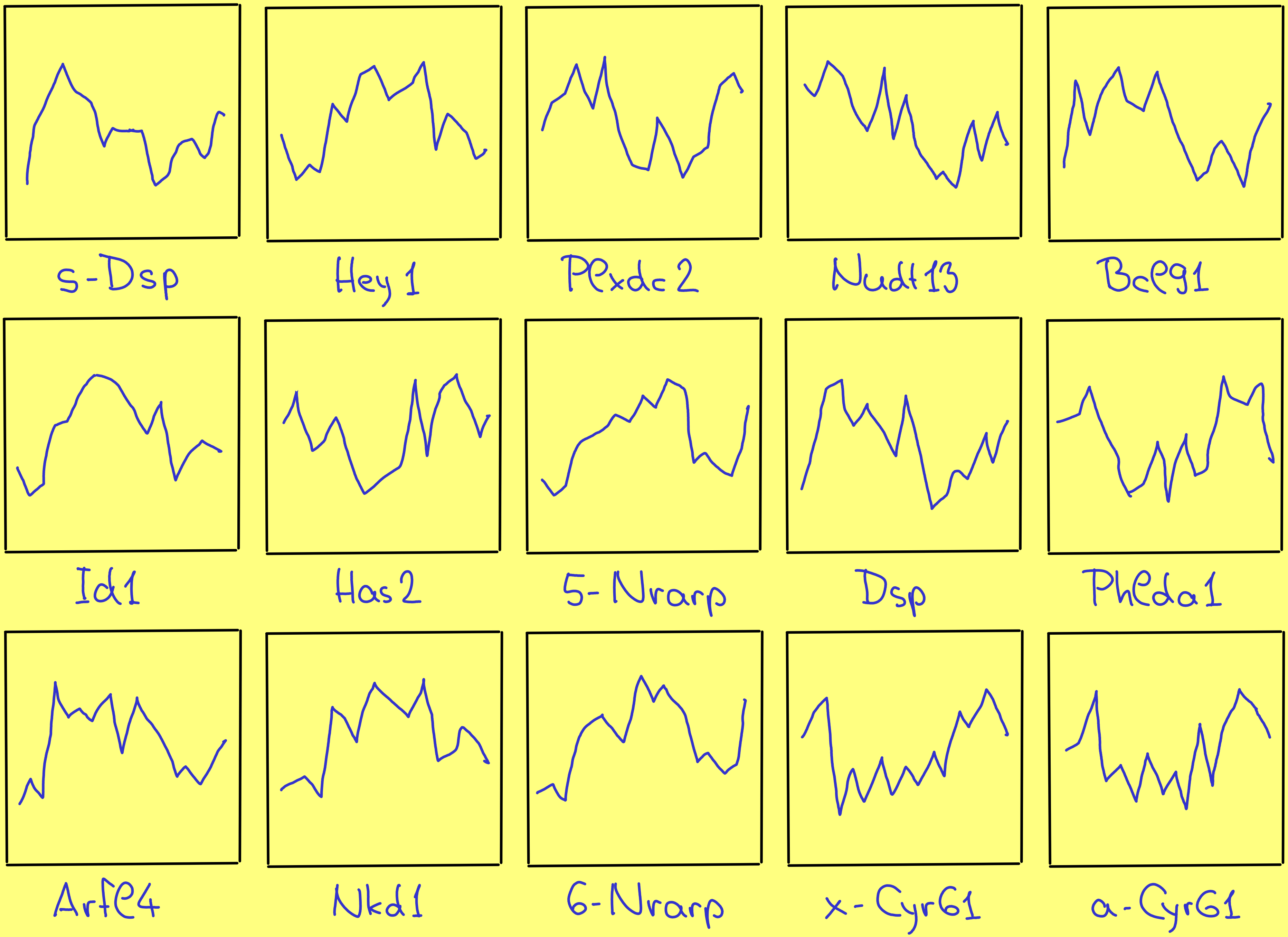
... time series microarray analysis



II.3 Rhythmic Gene Expression



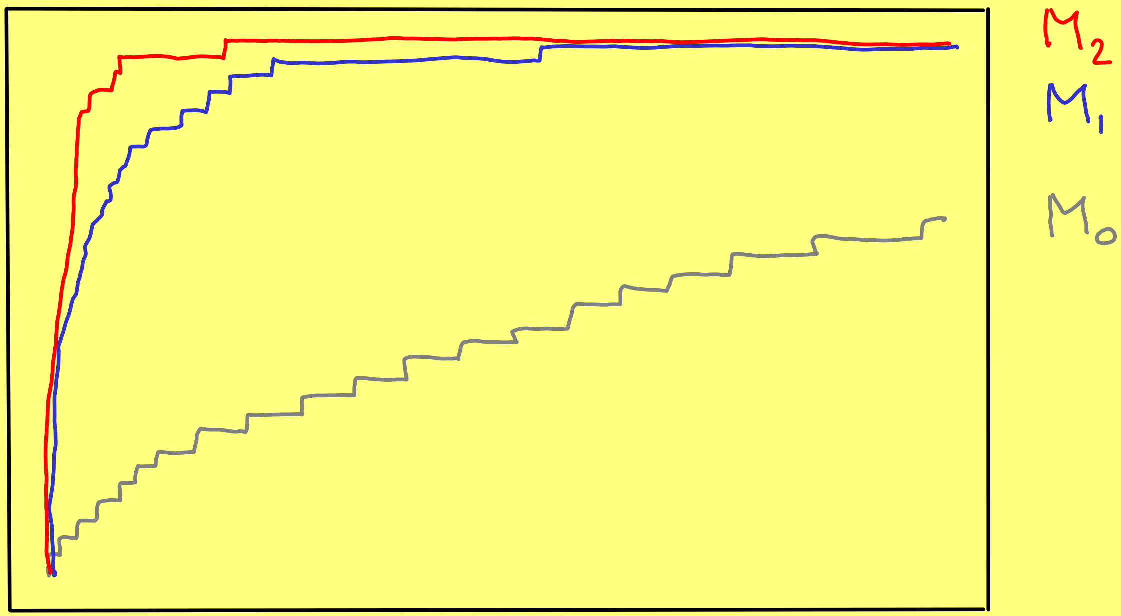

Dkk1
Tnfrsf19
Hes1
Axin2
Hspg2
Myc
Hes5
Dact1
Sp5
Efna1
Bcl2l11
a-Tfnrsf19
Lnfg
Spry2
Klf10


s-Dsp
Hey1
Plxdc2
Nudt13
Bcl91
Id1
Has2
5-Nrarp
Dsp
Phlda1
Arfl4
Nkd1
6-Nrarp
x-Cyr61
a-Cyr61

II.3 Rhythmic Gene Expression

yield of 30 biologically confirmed genes

ROC Curves



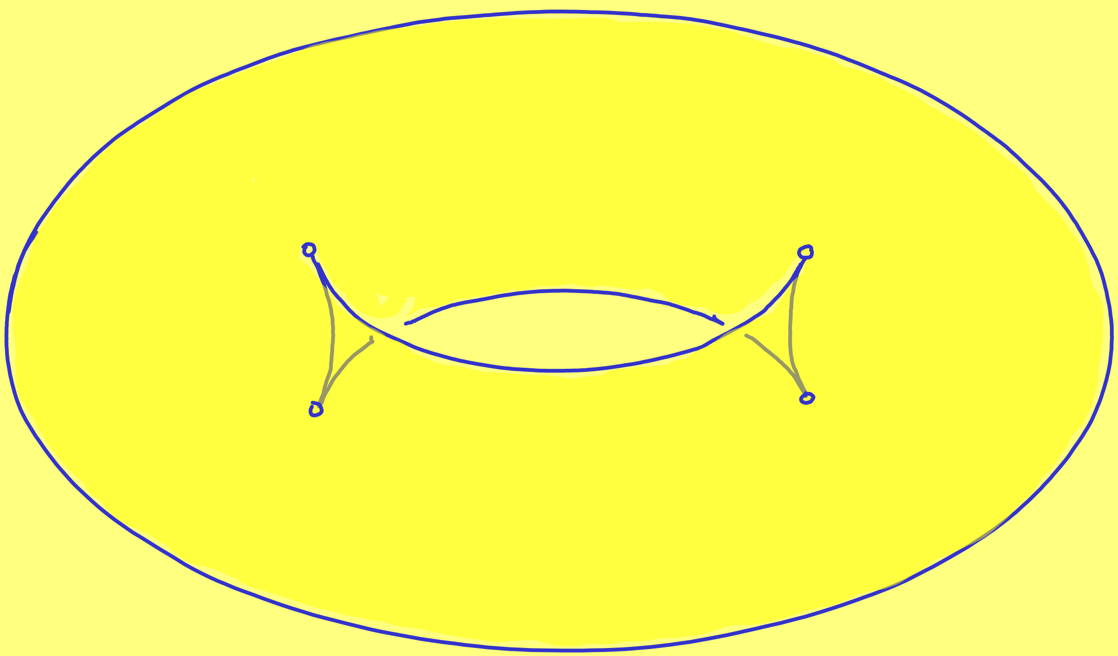
	M_0	M_1	M_2	M_3	M_4	max
Area	4.42	9.75	10.18	10.19	10.15	10.50

I Persistent Homology

II L_p-stability and Somites

III Contour Stability

III.1 The Contour

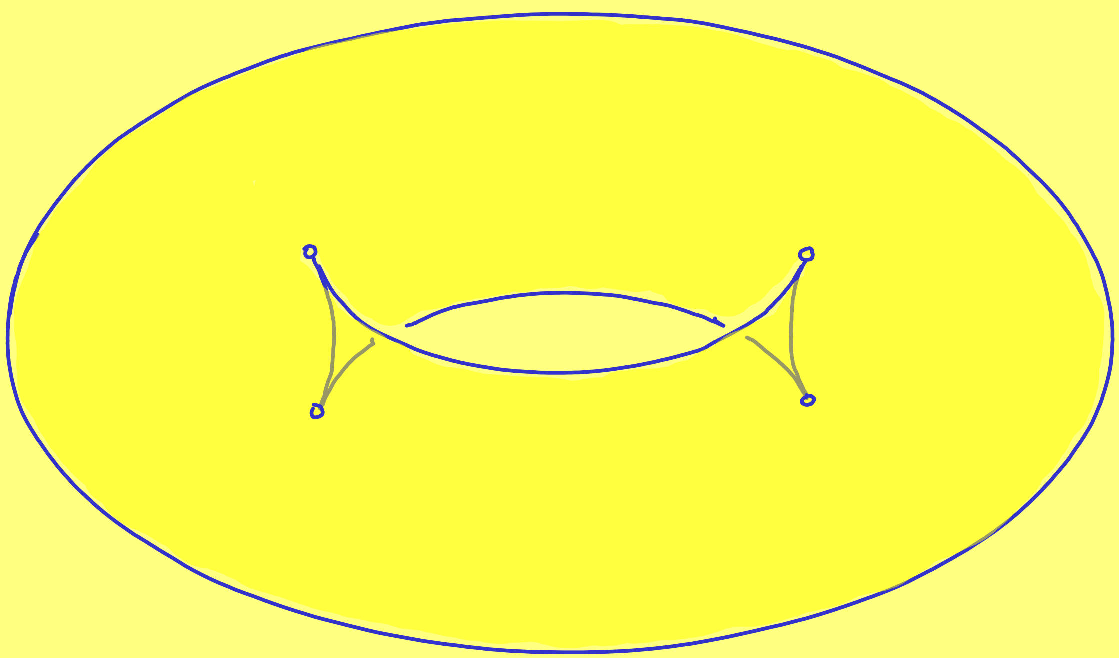


III.1 The Contour

$$f : \mathbb{M} \to \mathbb{R}^2$$

generic smooth

compact, orientable 2-mfld w/o boundary



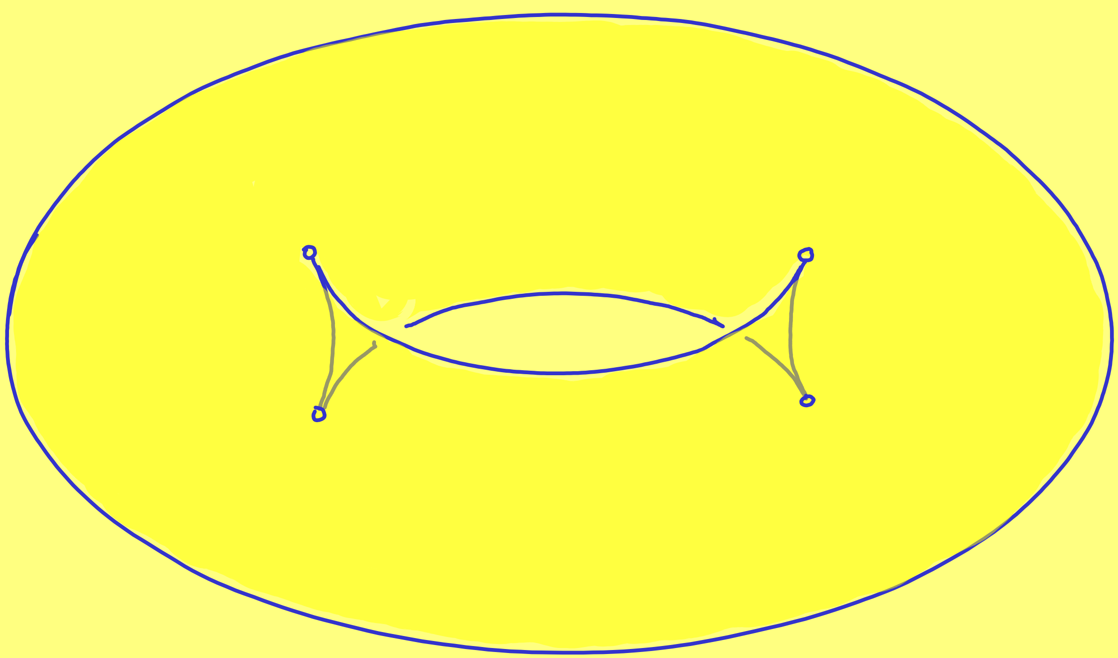
III.1 THE CONTOUR

$f : \mathbb{M} \to \mathbb{R}^2$

generic smooth

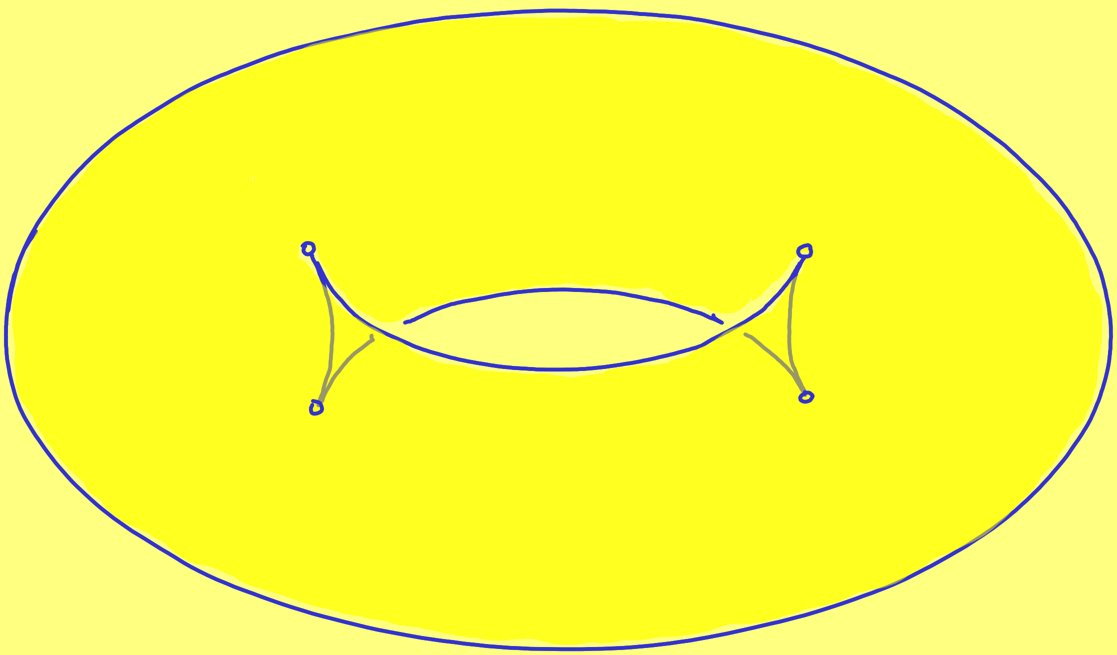
compact, orientable 2-mfld w/o boundary

Contour(f) =
set of critical values



III.2 A Perturbation

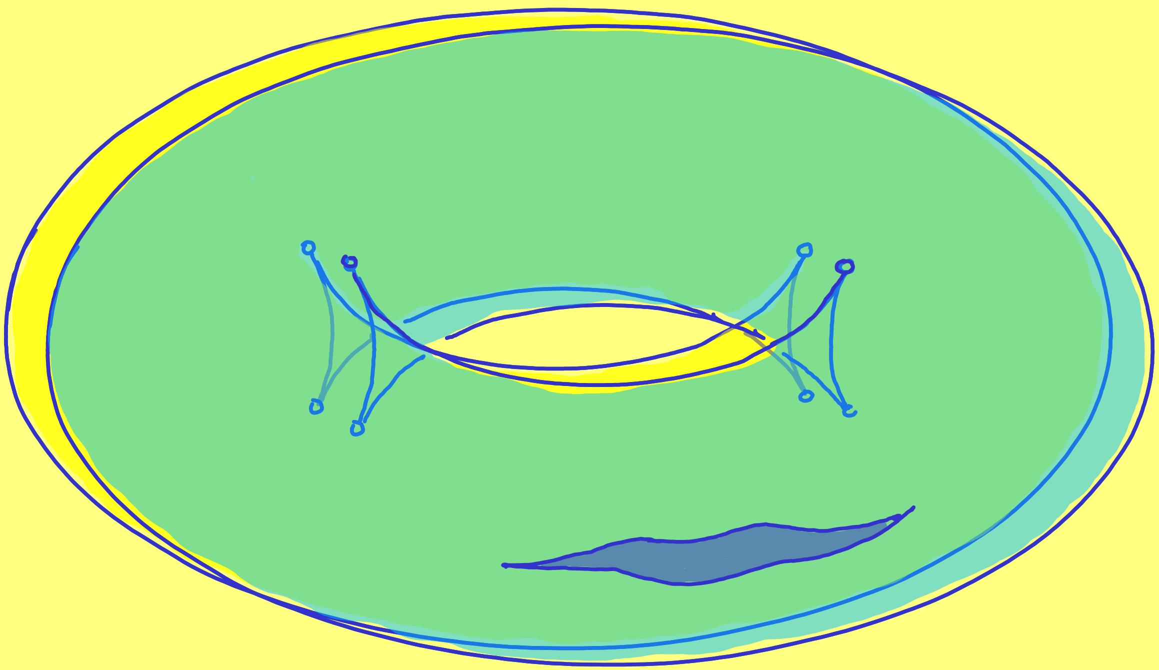
$f : \mathbb{M} \to \mathbb{R}^2$



III.2 A Perturbation

$$f, g : \mathbb{M} \to \mathbb{R}^2$$

$$\varepsilon = \max_{x \in \mathbb{M}} \| f(x) - g(x) \|_2$$

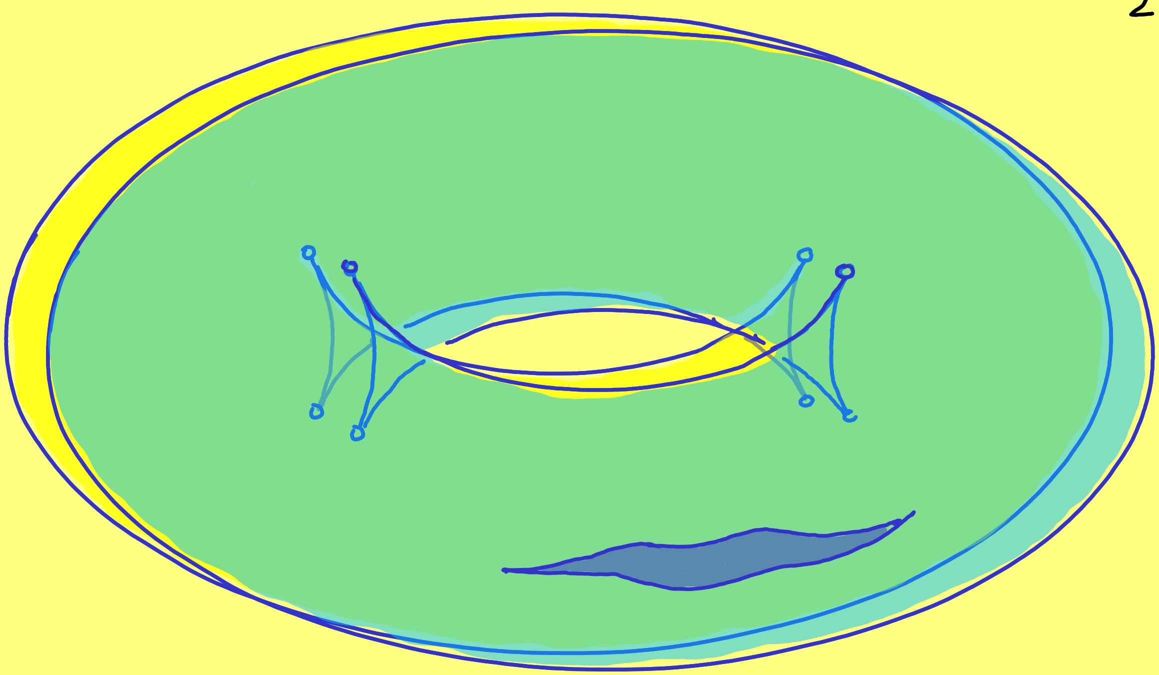


III.2 A Perturbation

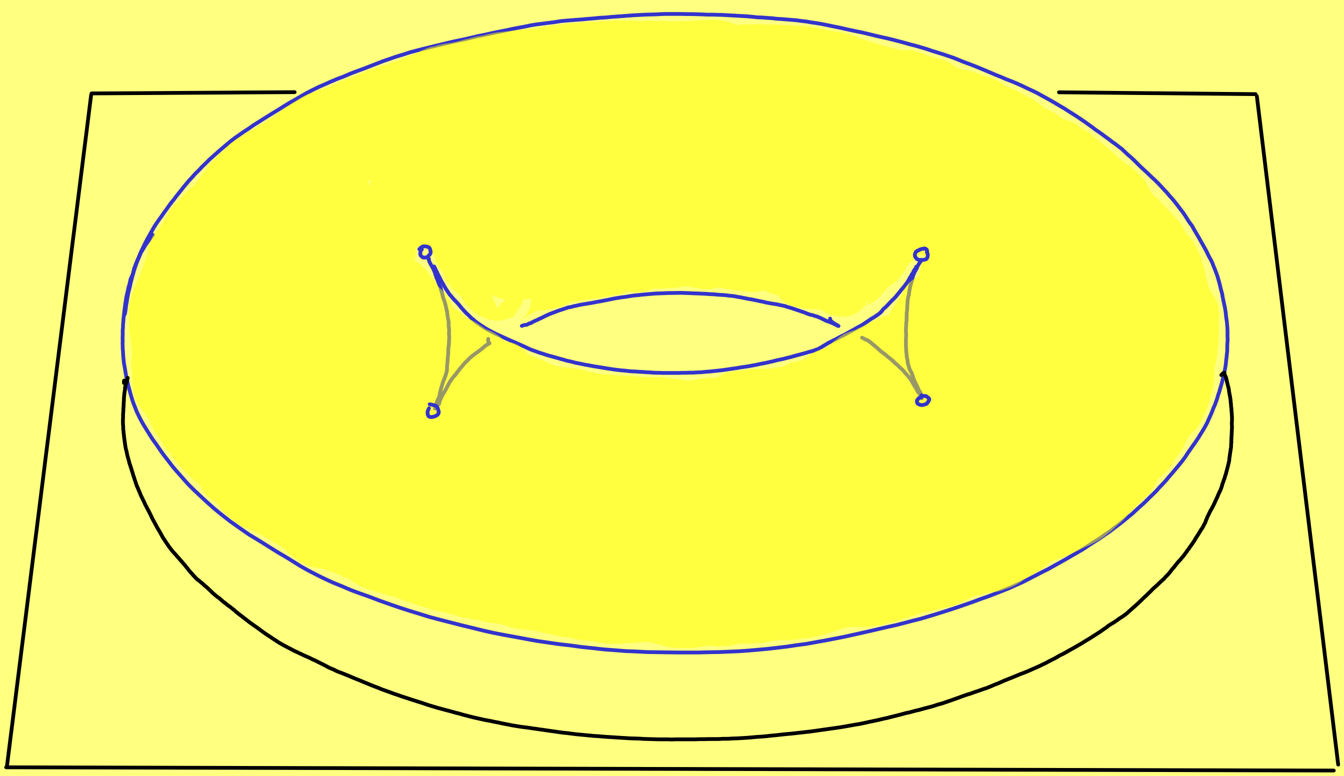
$$f, g : \mathbb{M} \to \mathbb{R}^2$$

$$\varepsilon = \max_{x \in \mathbb{M}} \| f(x) - g(x) \|_2$$

1. close contour lines;
2. thin creases.



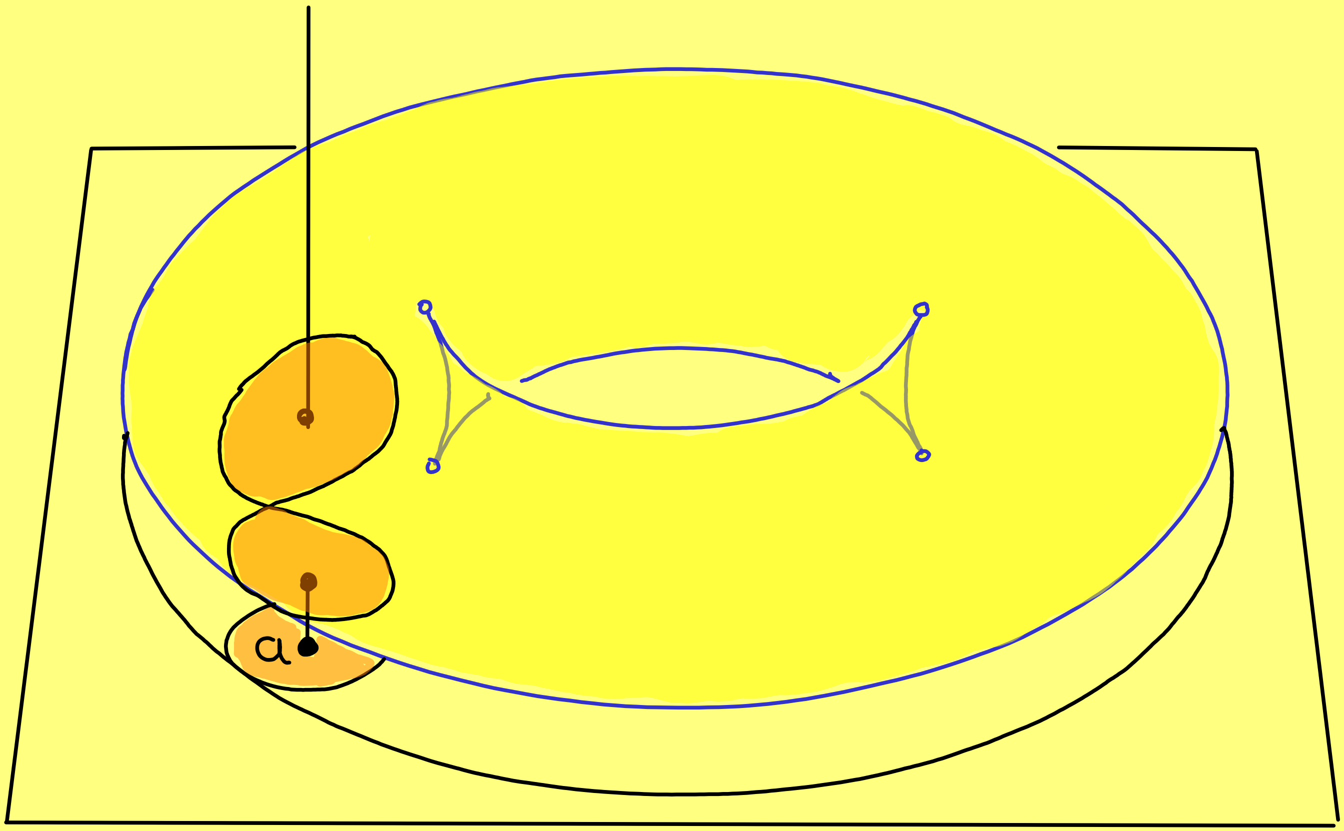
III.3 Distance Function



III.3 Distance Function

$f_a : \mathbb{M} \to \mathbb{R}$ defined by

$$f_a(x) = \| f(x) - a \|_2$$



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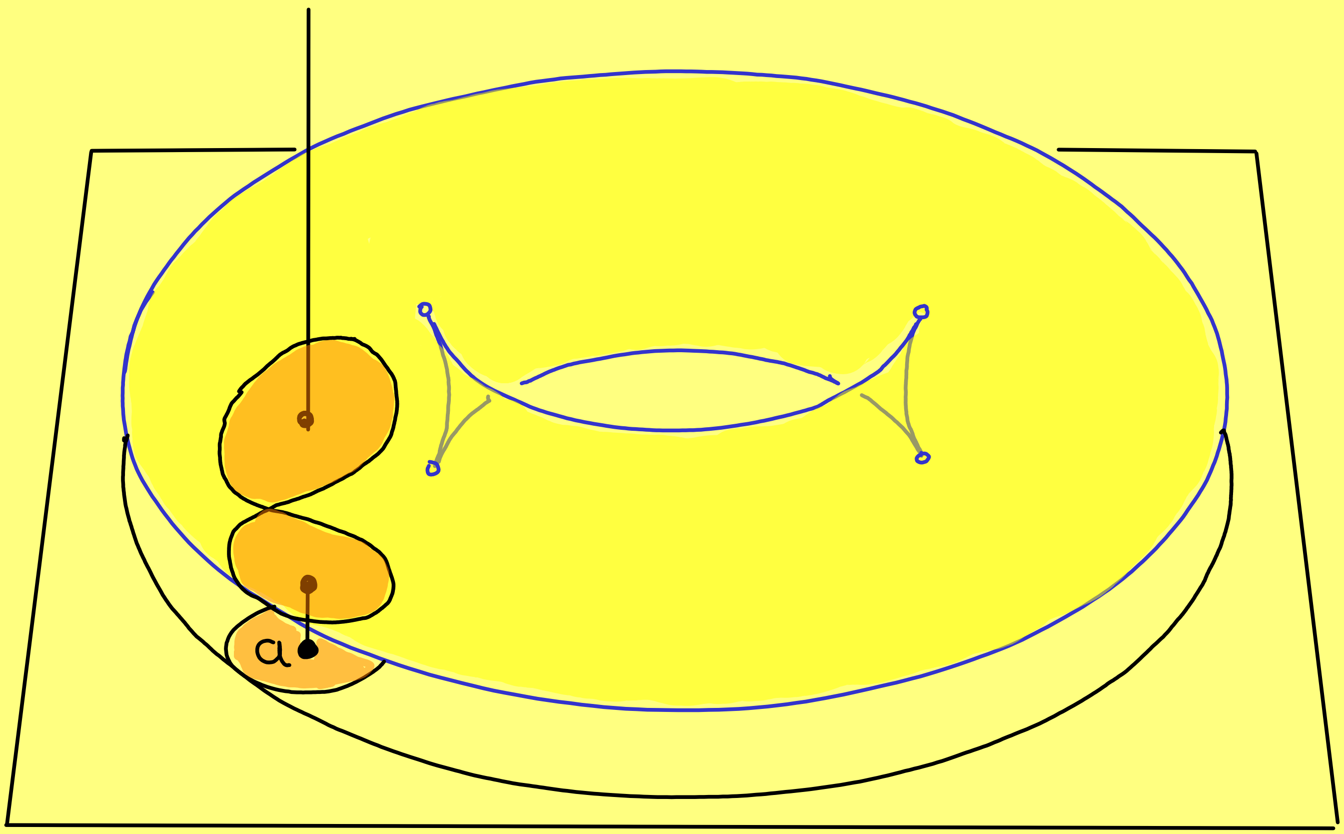
$f_a : \mathbb{M} \to \mathbb{R}$ defined by

$$f_a(x) = \| f(x) - a \|_2$$

sublevel set is

$$\mathbb{M}_r(a) = f_a^{-1}[0, r]$$



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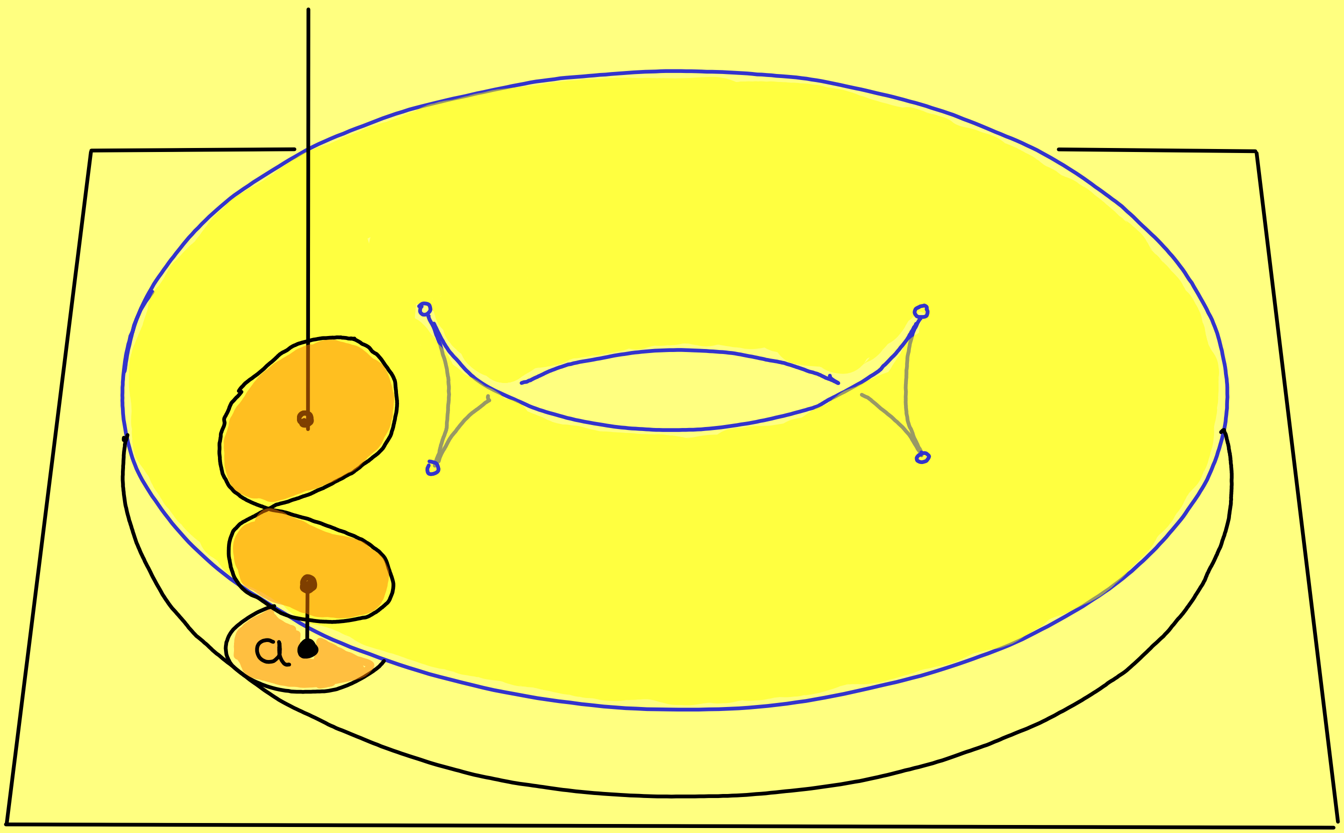


sublevel set is

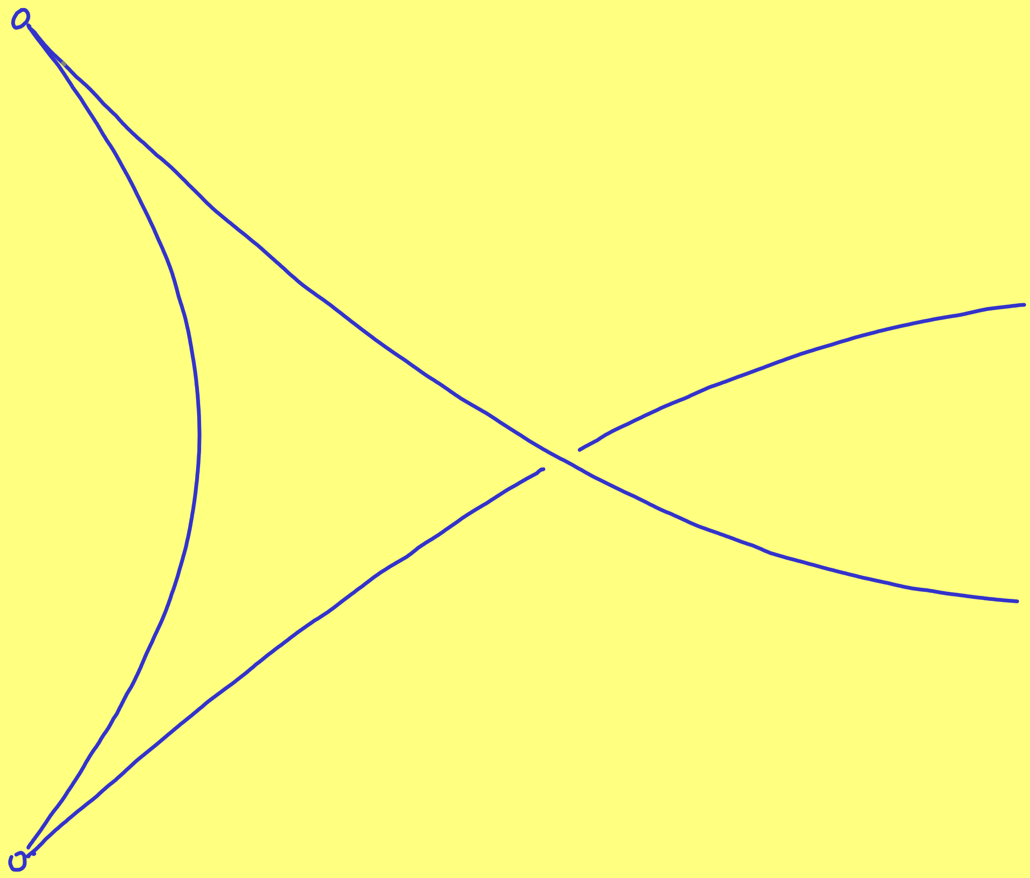
$$\mathbb{M}_r(a) = f_a^{-1}[0, r]$$

x crit. for f_a

$\Downarrow$

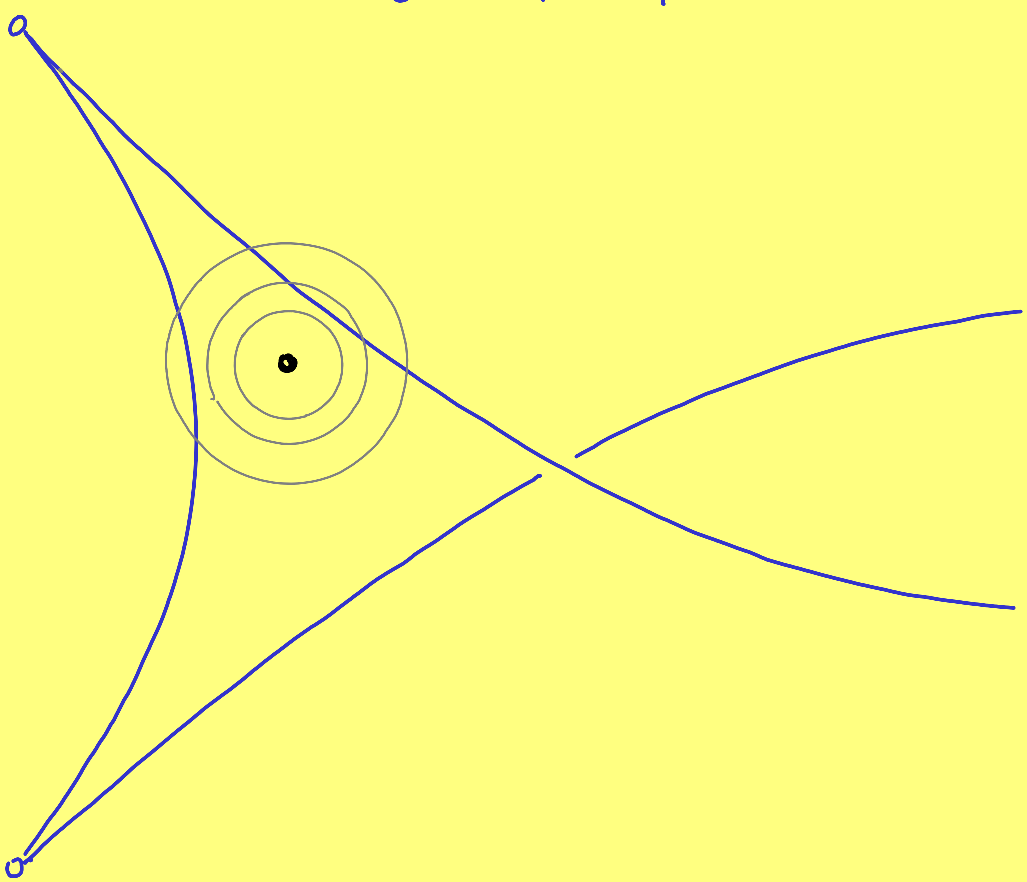
x crit. for f

III.4 Degree



III.4 DEGREE

$\mathbb{F}_r = H_0(\mathbb{M}_r(a))$ indexed by radius



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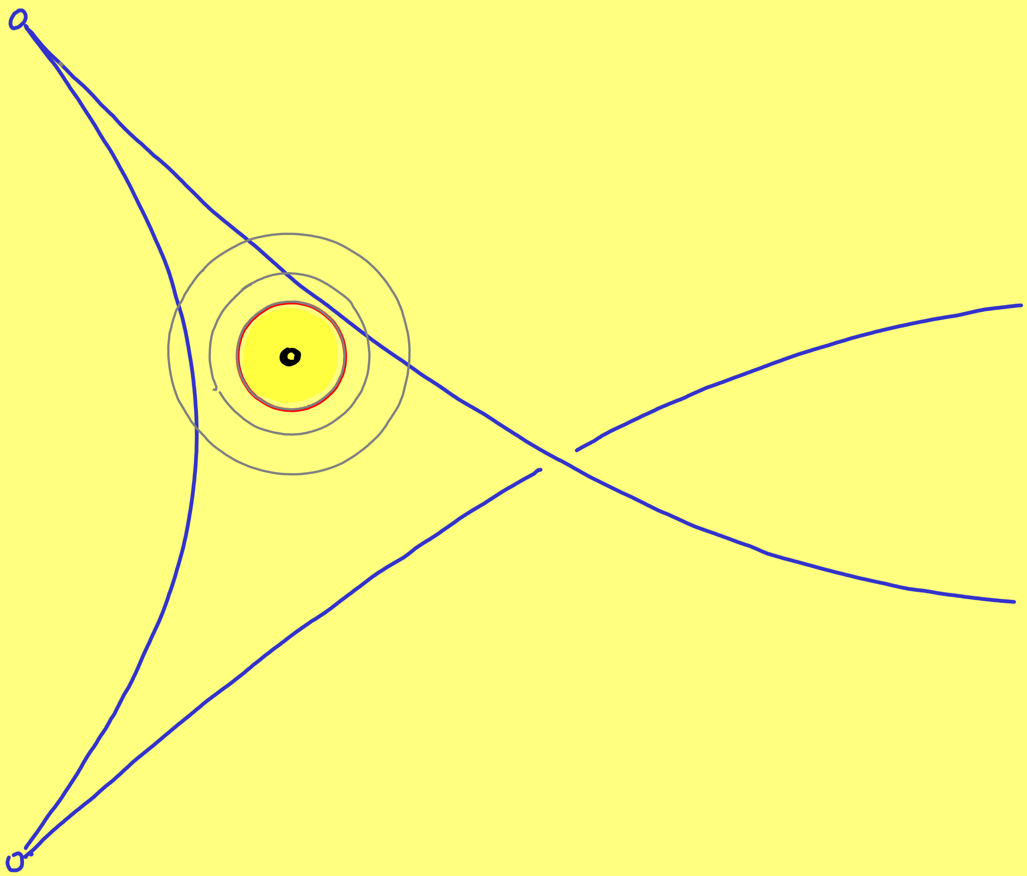
$F_r = H_0(\mathbb{M}_r(a))$ indexed by radius



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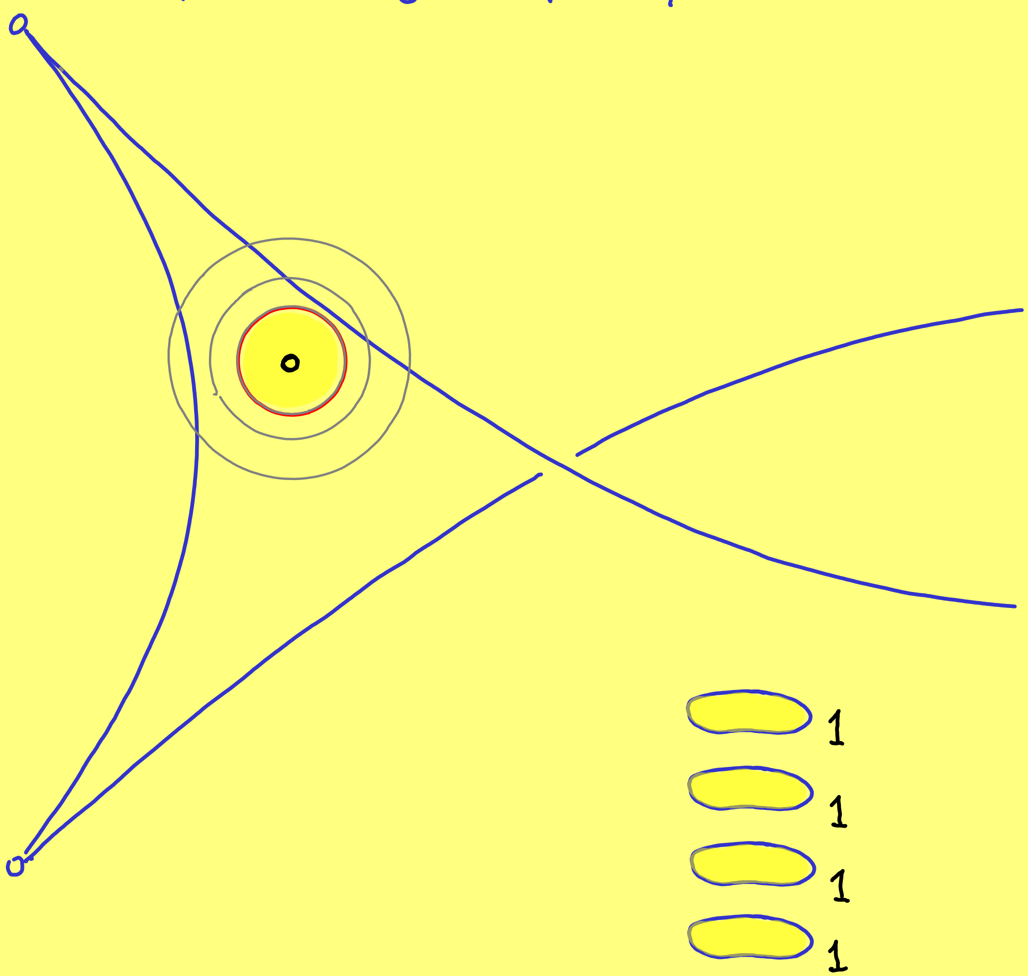

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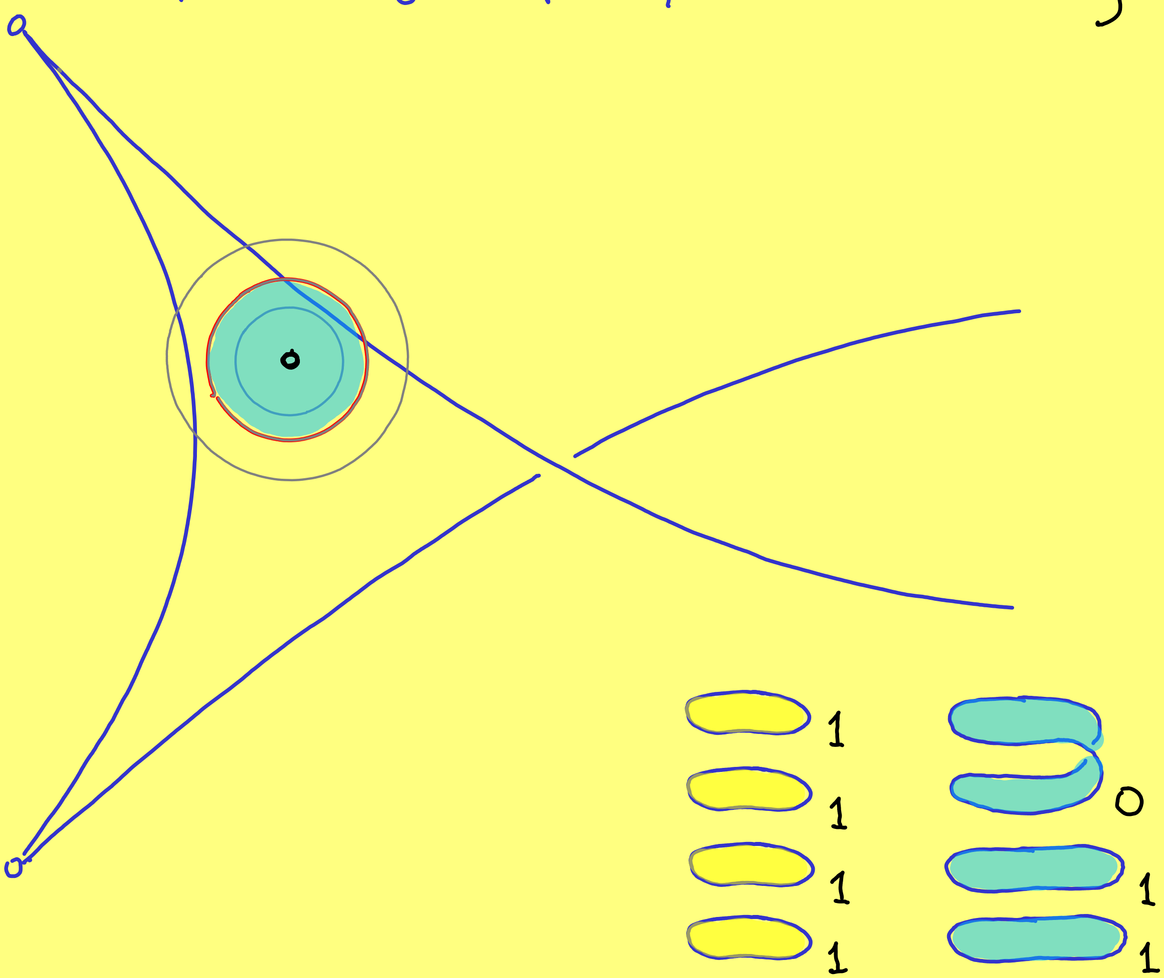

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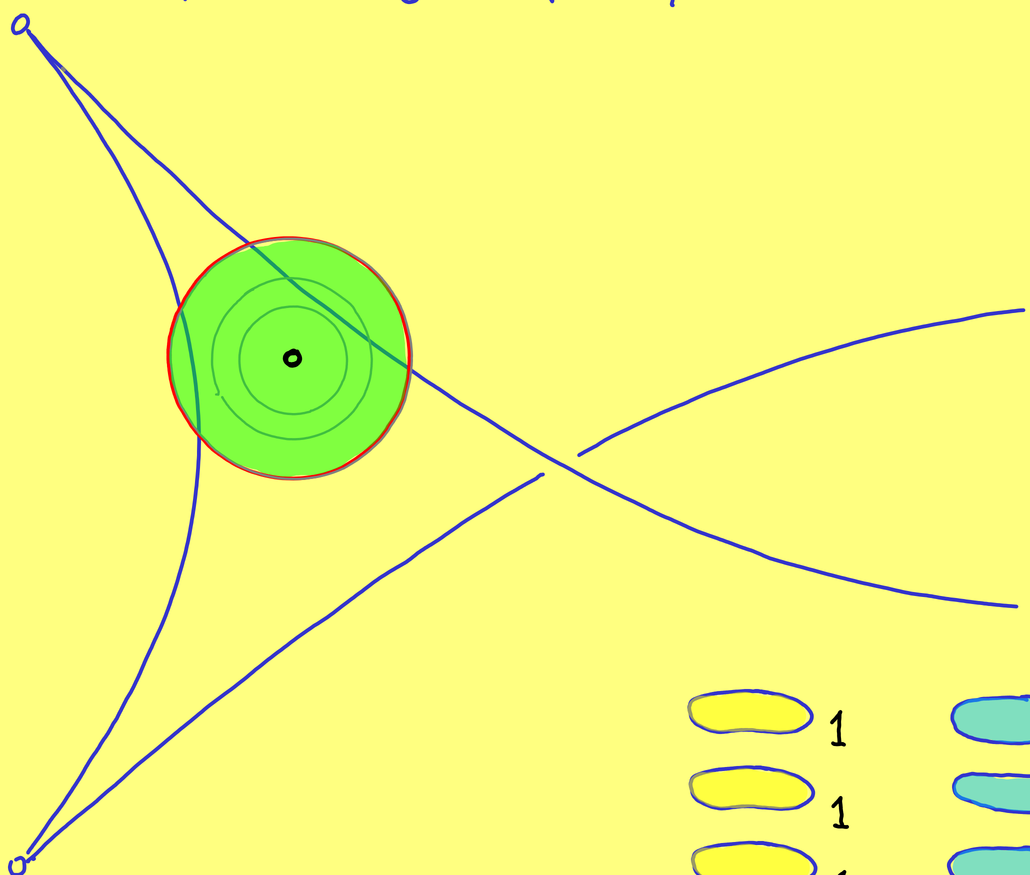

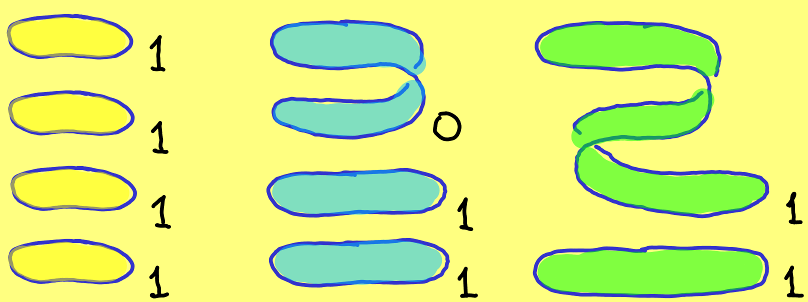

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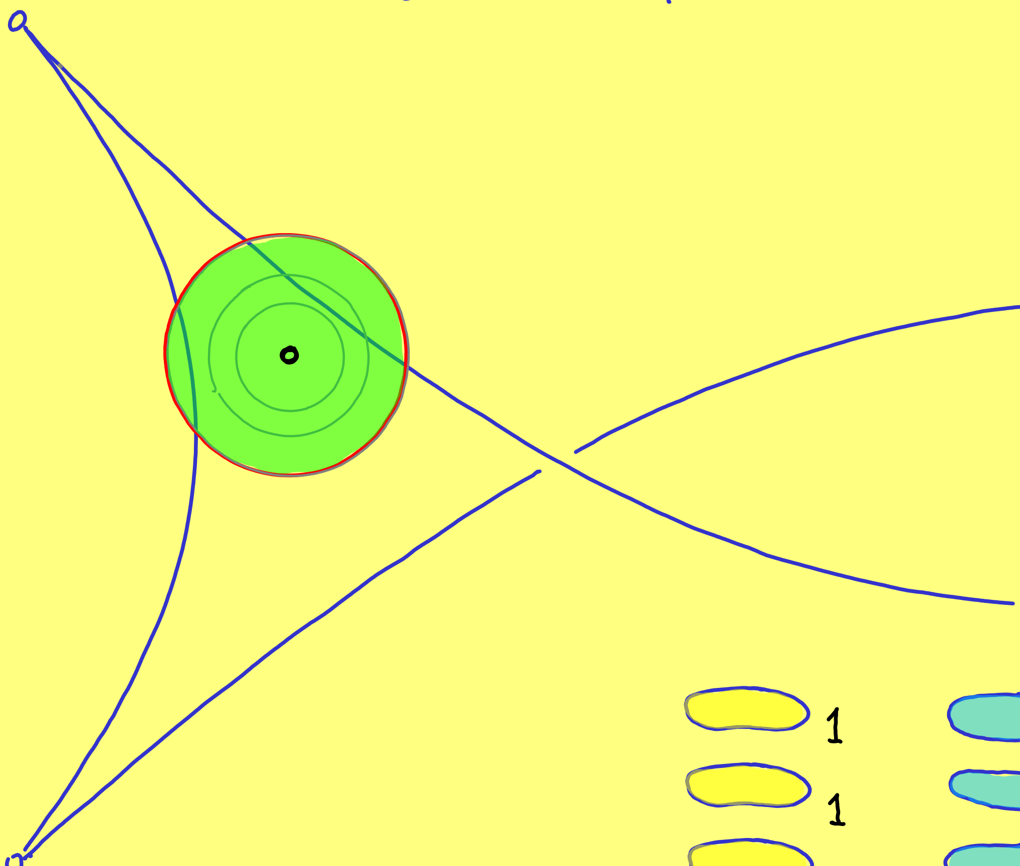
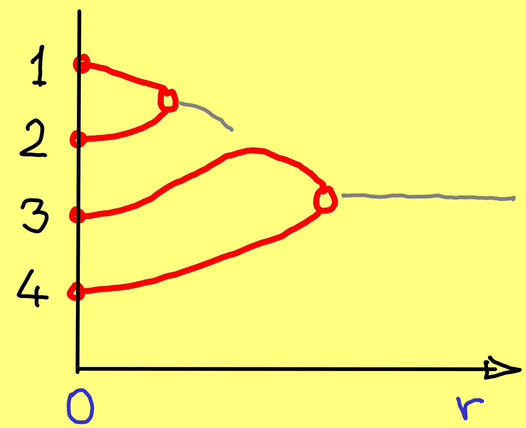
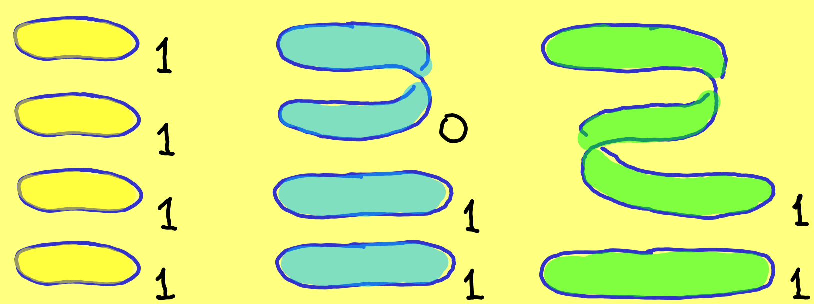
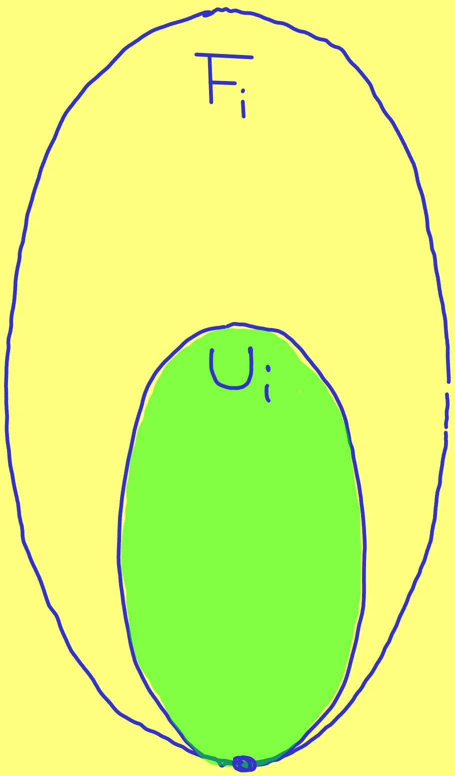


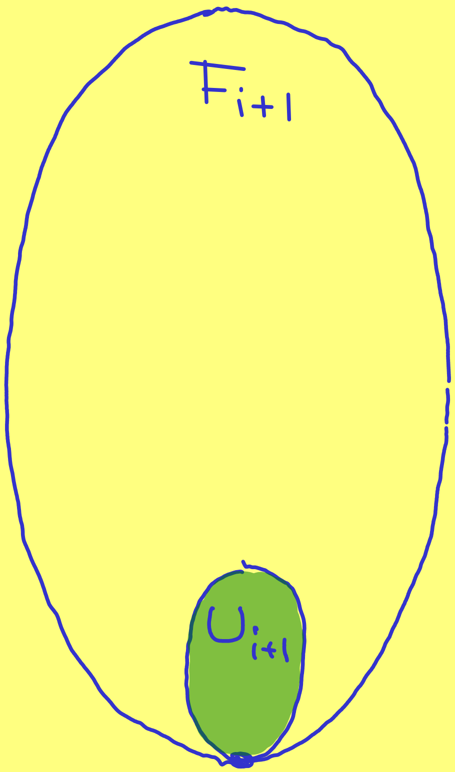




III.5 WELL GROUP

$U_i \subseteq F_i$ well subgroup of $H_0(\mathbb{M}_r(a))$





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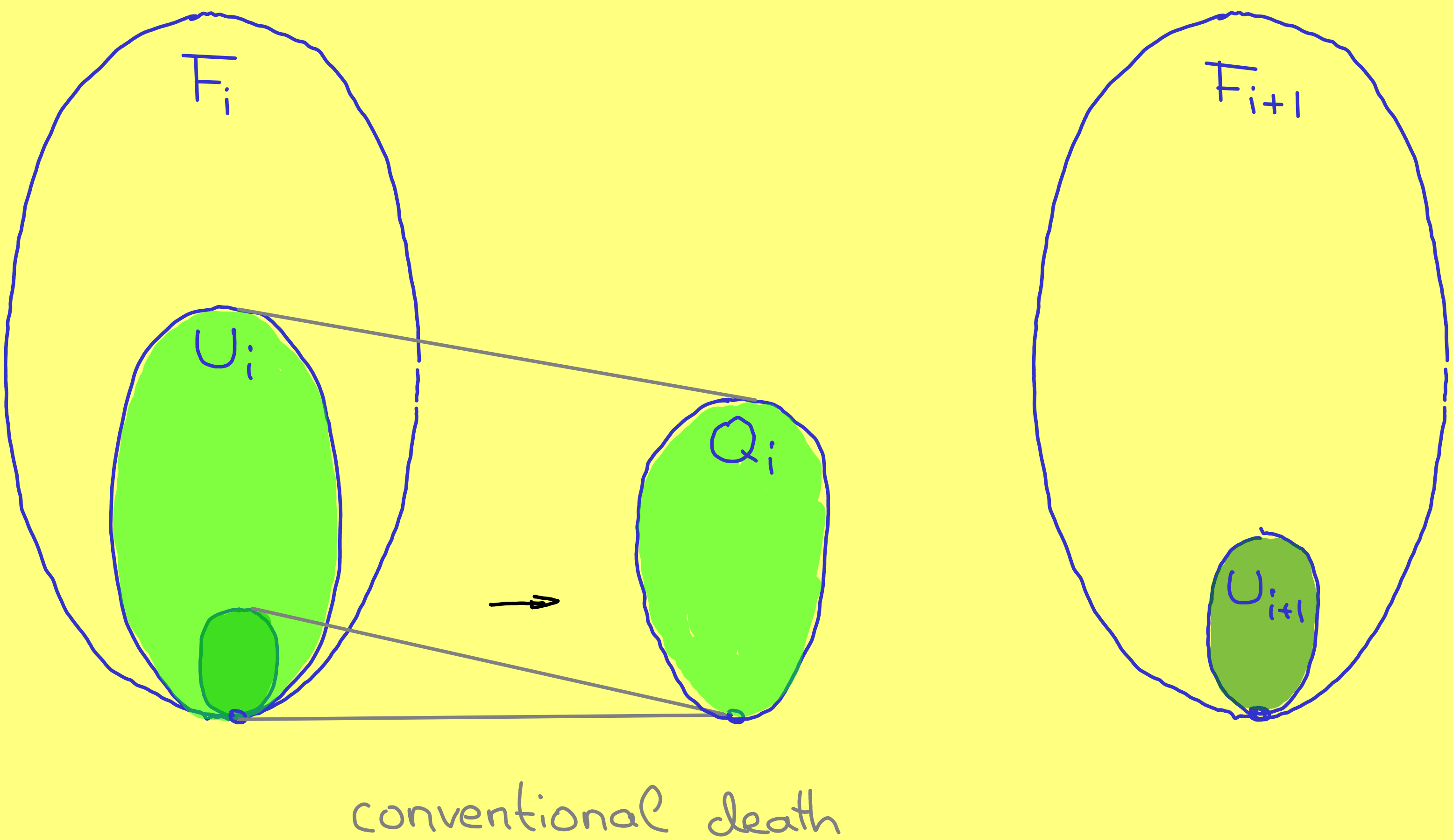

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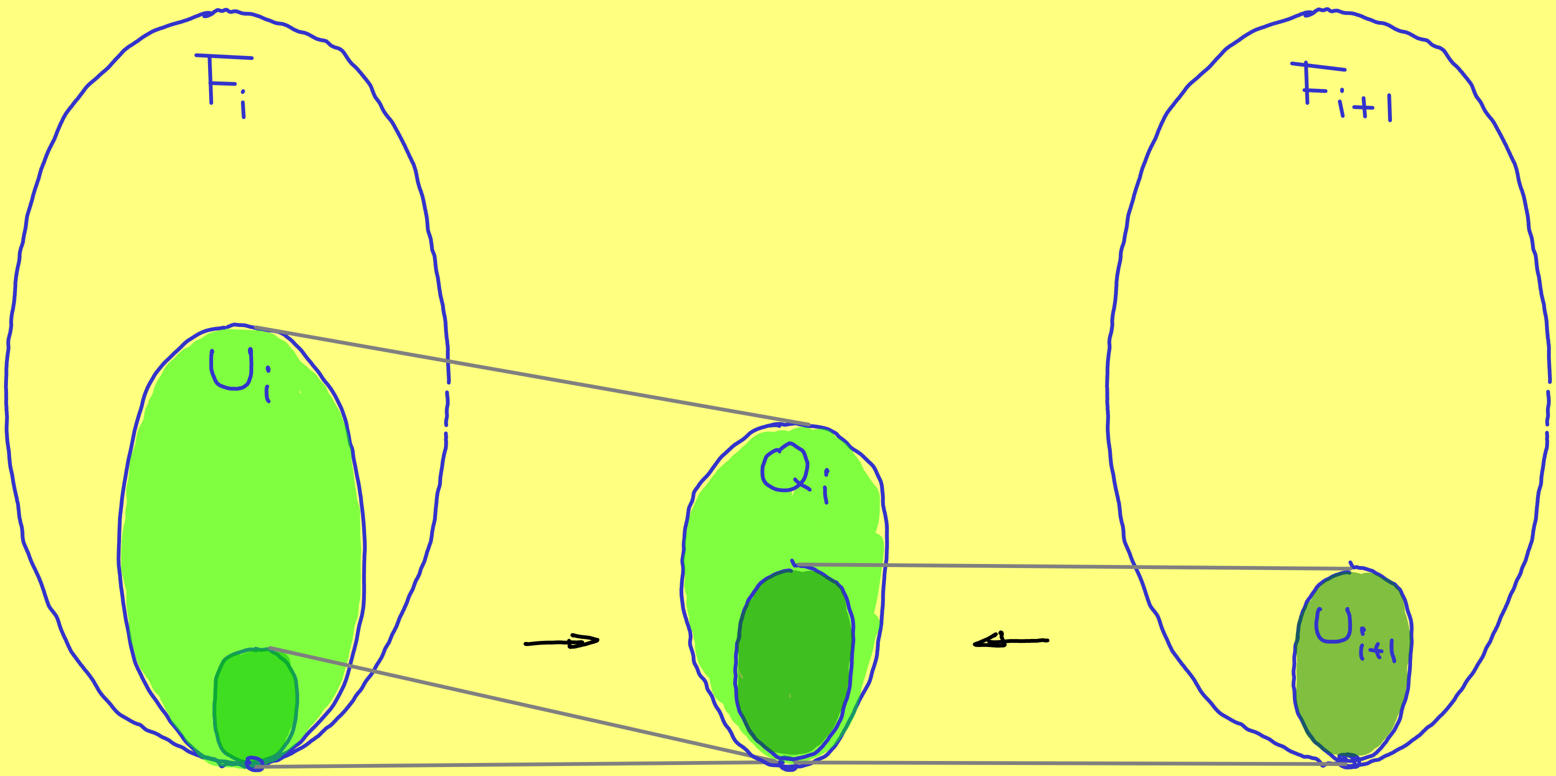

unconventional death

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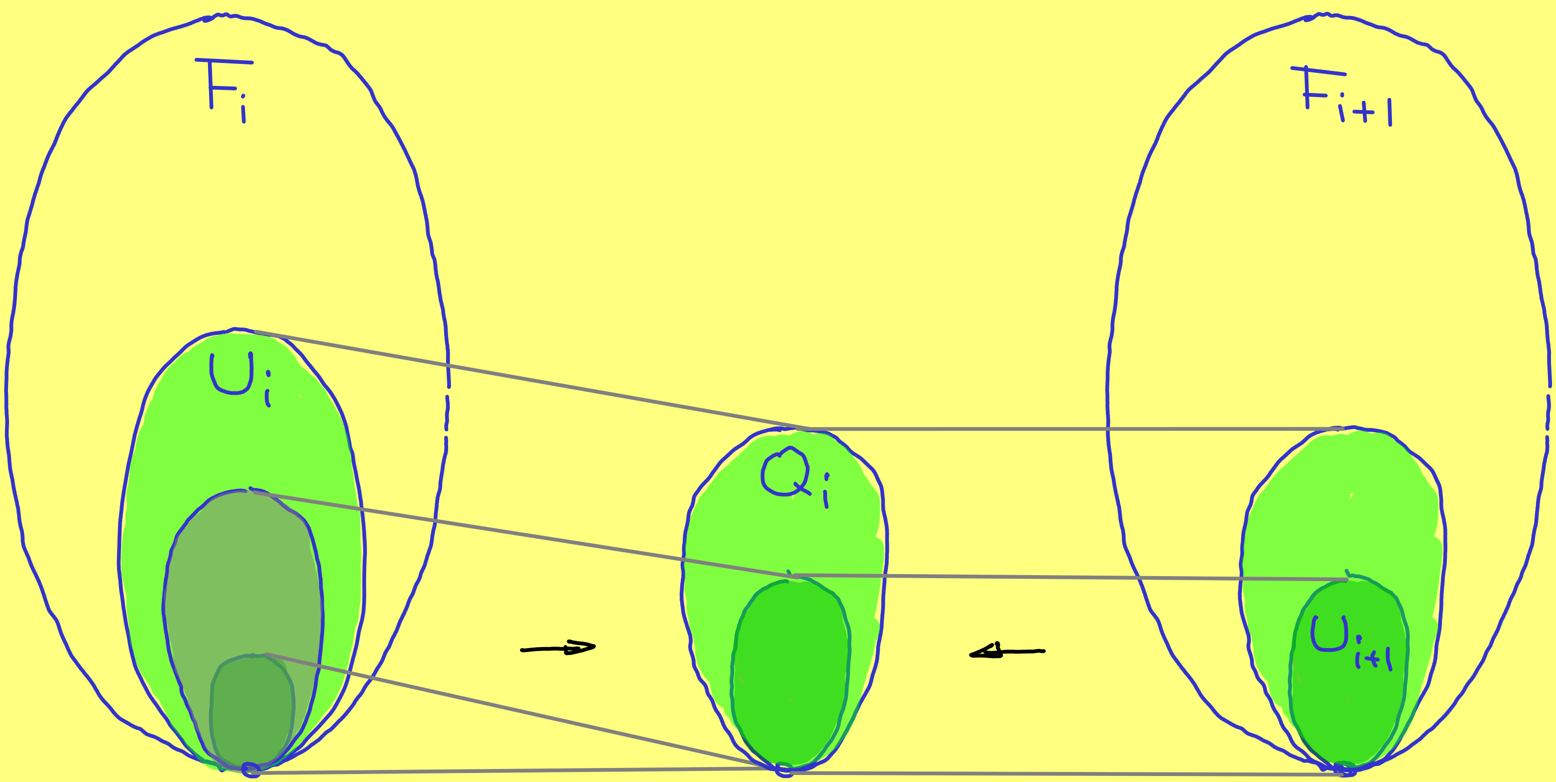
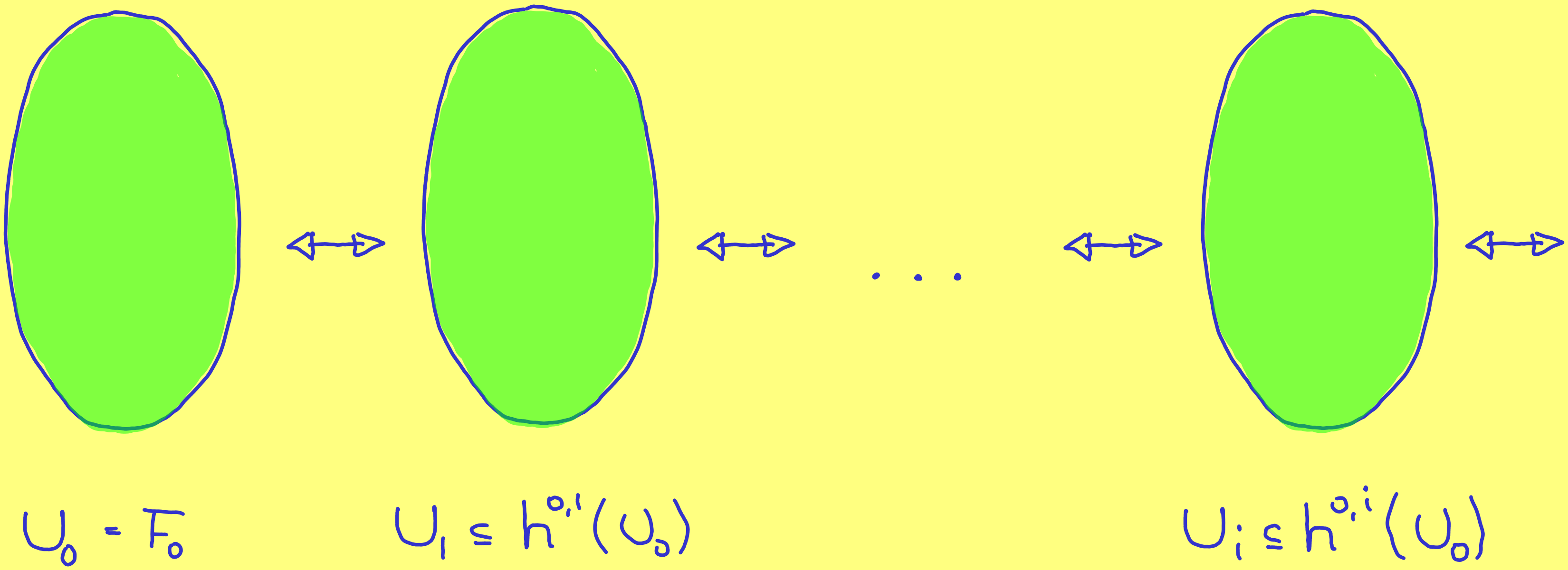
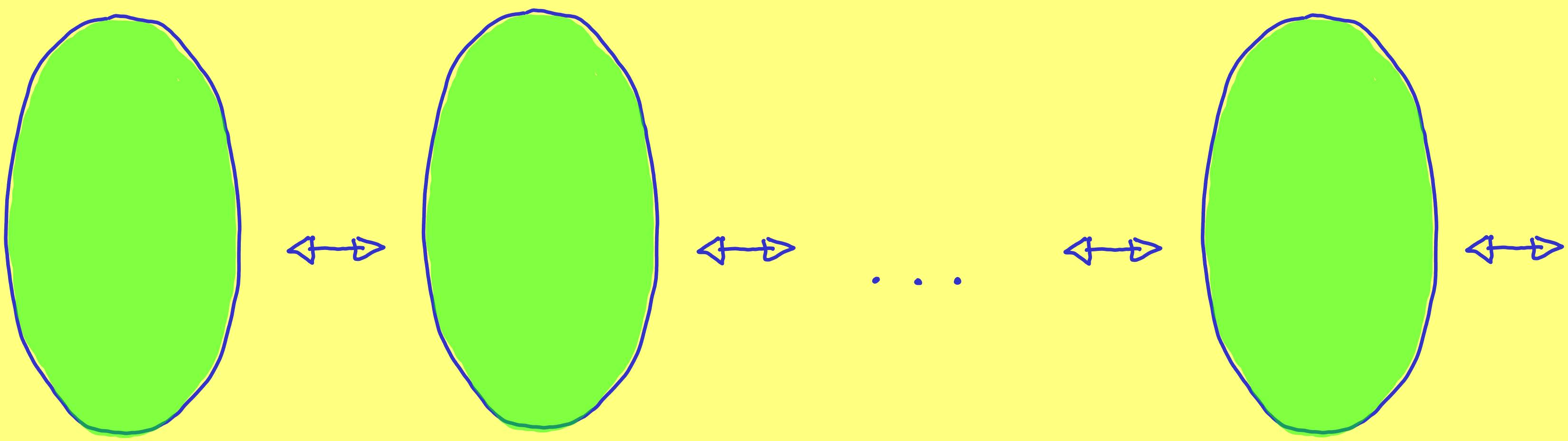


III.6 Zigzag Module



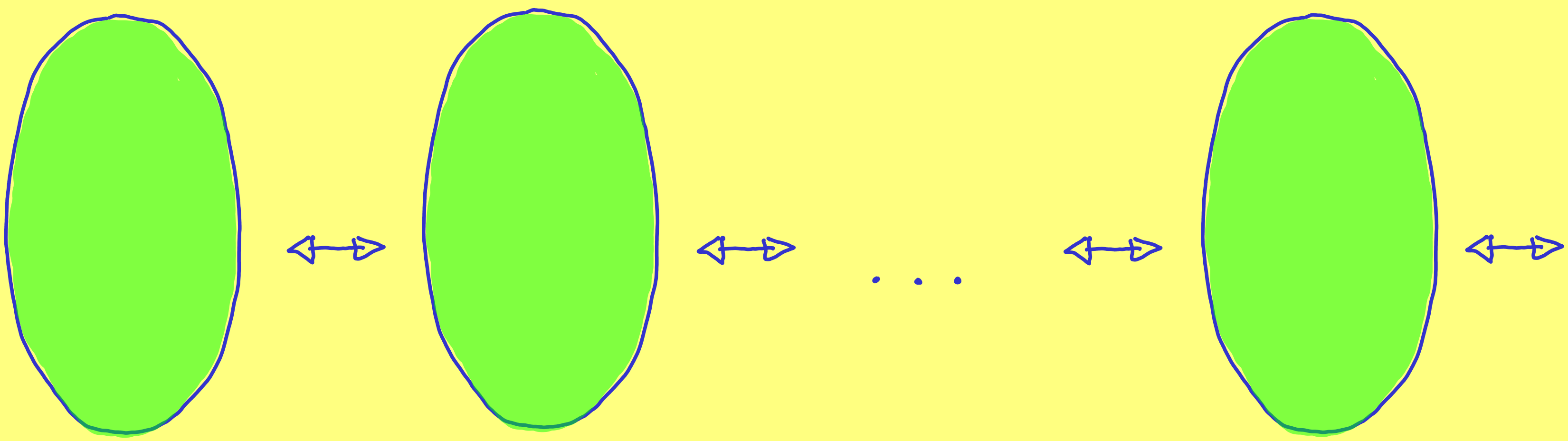
III.6 Zigzag Module



$U_0 = F_0$ $\qquad$ $U_1 \subseteq h^{0,1}(U_0)$ $\qquad$ $U_i \subseteq h^{0,i}(U_0)$

persistence diagram exists

III.6 Zigzag Module

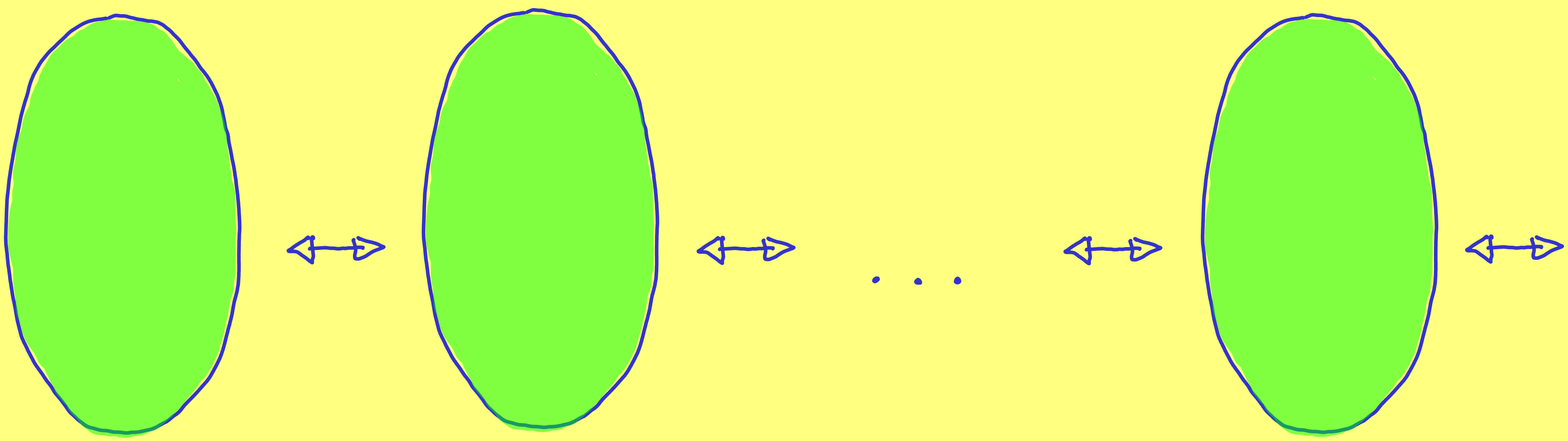


$U_0 = F_0$ $\quad U_1 \leq h^{0,1}(U_0)$ $\quad U_i \leq h^{0,i}(U_0)$

persistence diagram

exists
one-dimensional

III.6 Zigzag Module

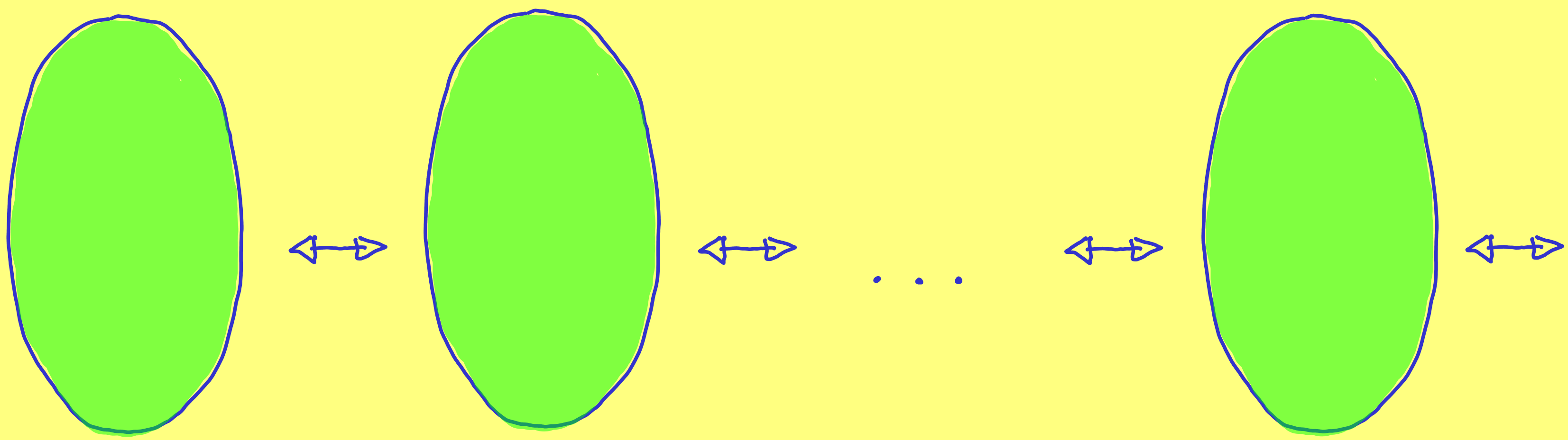


$U_0 = F_0$ $\qquad$ $U_1 \subseteq h^{0,1}(U_0)$ $\qquad$ $U_i \subseteq h^{0,i}(U_0)$

persistence diagram

exists
one-dimensional
stable

III.6 Zigzag Module



$U_0 = F_0$ $\quad$ $U_1 \subseteq h^{0,1}(U_0)$ $\quad \ldots \quad$ $U_i \subseteq h^{0,i}(U_0)$

persistence diagram

exists
one-dimensional
stable $\Longrightarrow$ contour stable

THANK YOU