Partitioning 2D Images into Prototypes of Slope Region

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Abstract. A gray scale digital image can be represented as a 2.5D surface where the height of the surface corresponds to the gray value of the respective pixel. Analysis of the gray scale image can be efficiently done by exploiting the properties of the plane graph embedded in the 2.5D surface. The vertices of the graph can be easily categorized into critical and non-critical points by use of Local Binary Patterns (LBPs). Well defined graph operations such as contraction and removal of edges are used to eliminate the non-critical points and preserve the critical points thereby reducing the size of graph. In this process, it is important to preserve the structural and topological properties of the regions of a gray scale image. After analysing the topological properties of a well composed image, we provide two prototypes of the slope region and the necessary conditions for their existence. Also we prove that every slope region conforms to either of the two prototype. Conversely the prototypes may be used to generate an image with a required topological properties.

1 Introduction

Exploiting the surface properties by its representation using the surface elements (viz. local maximum, local minimum, etc.) and simultaneously preserving the structural (topological) properties has been a classical problem in pattern recognition and image processing. It has various applications like multi-resolution image segmentation, image compression and so on. The surface elements are broadly classified into two types: critical points (maximum, minimum, saddle) and non-critical points (slope point). Cayley [4] and Maxwell [14] explored the critical points and slope lines of a surface in terms of earth's topography. A century later, Lee [13] came up with a graphical representation of the surface and enumerated different possible configurations of the critical points in a Morse function.

Identifying the critical points in the neighborhood graph of a digital image was described in [5] using Local binary Patterns (LBPs) which eliminated the computation of differentiation. An extension, Cerman *et al.* [6] provided an algorithm for multi-resolution image segmentation using the graph pyramid which is a stack of reduced graphs. Wei in [15] uses an approach to construct a hierarchical structure similar to the graph pyramid called 'super-pixel hierarchy' for multi-resolution image segmentation. Edelsbrunner *et al.* [8] discuss the construction of

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a hierarchy of increasingly smaller Morse-Smale complexes to decompose a piecewise linear 2-D manifold. In [11,12] authors provide definitions of slope regions and slope complex which generalizes Morse-Smale complexes and enumerates the different configurations of the slope regions formed by critical points. [2] deals with a prototype of a slope region and counts the number of slope regions at a given level of the graph pyramid. In [1] authors describe the necessary and sufficient conditions for merging slope regions in the region adjacency graph (RAG) and its dual the boundary adjacency graph (BAG).

After recalling basic definitions related to the topological aspects of digital images (Sect. 2), we extend our previous work [11] in several ways. In Sect. 3 we introduce a new prototype for the slope region. Section 4 is devoted to the description of holes and two different ways they are attached to the boundary of the slope region. In Sect. 5 we prove that any digital image can be partitioned into slope regions of one of the two prototypes. Finally, in Sect. 6 we explain the necessary conditions for the existence of a saddle point on the boundary of the slope region.

2 Basic Definitions and Formation of a Slope Region

A digital image P can be visually perceived as a sampled version of a geographical terrain model which is a continuous surface. The sampling frequency to choose the samples should satisfy the Nyquist criterion for the minimum distance between any two critical points. The digital image P can be efficiently represented by a dual pair of plane graphs. The region adjacency graph (RAG) $G = (V, E)$ is formed by vertices $v \in V$ corresponding to pixels $p \in P$ connected to the four adjacent neighbors by edges $e \in E$. The dual of the RAG is the boundary adjacency graph (BAG) $\overline{G} = (\overline{V}, \overline{E})$ where every vertex $\overline{v} \in \overline{V}$ of BAG corresponds to a face formed by the intersection of the boundary segments in the RAG G and edges $\overline{e} \in \overline{E}$ of the BAG correspond to the boundary separating the faces in the RAG G [7, Sect. 4.6]. The gray value of the pixel p is visually conceived as the height of the surface and it is denoted by $g(p) = g(v)$ where v is the vertex corresponding to p. There are two operations to build a graph pyramid [10]: contraction and removal of edges in the graph. Contraction of an edge [7, Sect. 1.7] in G will result in merging the corresponding two pixels connected by the respective edge. This is equivalent to the removal operation in the BAG $\overline{G}$. Duality imposes a one-to-one correspondence between the edges of the RAG G and of its dual the BAG $\overline{G}$. The removal of an edge $(v, w) \in E$ disconnects the two vertices v and w and merges the two faces which is equivalent to contract $\overline{e} \in \overline{E}$ in $\overline{G}$.

By successively contracting and removing edges, we form a stack of progressively reducing planar graphs $(G_k, \overline{G_k}), k \in \{0, 1, \ldots, n\}$ where each graph G_{k+1} is smaller than the graph G_k [3,9,10]. The base level of the graph pyramid is the neighborhood graph or RAG G_0.

Definition 1. *The* ***orientation of an edge*** $(v, w) \in E$ ***in the RAG*** $G = (V, E)$ *is directed from vertex* $v \in V$ *to vertex* $w \in V$ *iff* $g(v) > g(w)$*, otherwise edges are not oriented.*

The edge $e \in E$ connecting two vertices $v, w \in V$ with $g(v) = g(w)$ is non-oriented. Note that we define the orientation of edges by considering only the gray values as a feature of an image. The theory stated in this paper remains valid for higher dimensional feature vectors provided that their ordering is defined.

Now using the orientation of an edge incident to a vertex, we can categorize a vertex into a local maximum, local minimum, saddle or a slope point.

Definition 2. *A vertex* $v \in V$ *is a* ***local maximum*** $\oplus$ *if all the edges incident to* v *are oriented outwards.*

Definition 3. *A vertex* $v \in V$ *is a* ***local minimum*** $\ominus$ *if all the edges incident to* v *are oriented inwards.*

Definition 4. *A vertex* $v \in V$ *is a* ***saddle point*** $\otimes$ *if there are more than two changes in the orientation of edges when the edges incident on* v *are traversed circularly (clockwise or counter-clockwise direction).*

Definition 5. *A vertex* $v \in V$ *is a* ***slope point*** *if there are exactly two changes in the orientation of edges when traversed circularly (clockwise or counter-clockwise direction).*

Categorizing a vertex using orientation of edges incident to it is equivalent to that of LBP code. The LBP value of an outward oriented edges are encoded as 1 and inward orientated edges are encoded as 0. The LBP code of a vertex is formed by concatenating LBP values of the incident edges in clockwise or counter-clock wise direction. The LBP code of a maximum will consist of 1 only while the LBP code of a minimum will consist of 0 only. The LBP code of slope points will have exactly 2 bit switches and saddles will have more than 2 bit switches. By use of orientated edges, we avoid the calculation of derivatives and eigen-values of the Hessian matrix to categorize a vertex.

Definition 6. *A* ***path*** π *is a non empty sub-graph of* G*, consisting of an alternating sequence of vertices and edges* $\pi = v_1, e(v_1, v_2), v_2, \ldots, e(v_{r-1}, v_r), v_r$*. A path* $\pi(v_1, v_r)$ *is* ***monotonic*** *if all the oriented edges* $(v_i, v_{i+1}), i \in [1, r-1]$ *have the same orientation.*

Note: Paths with non-oriented edges are called *level curves*. A level curve can be part of a monotonic path.

A monotonic path $\pi(v_1, v_r)$ can be further extended by adding an edge oriented in the same direction as the direction of monotonic path $\pi(v_1, v_r)$. A monotonic path which cannot be further extended is called a **maximal monotonic path**. The end points of a maximal monotonic path will always be a local maximum and a local minimum. The definition of the monotonic paths is used to define the slope region which is the foundation for the rest of the paper.

Definition 7. *A face in a surface embedded plane graph G is a* ***slope region*** $\mathbb{S}$ *if all the pairs of points in the face can be connected by a continuous monotonic curve inside the face.*

Remark 1. The boundary $\delta\mathbb{S}$ of the slope region $\mathbb{S}$ is either a level curve or it can be decomposed into exactly two monotonic paths [11, Lemma 1].

Remark 2. **Property of a slope region:** Saddle points can only exist on the boundary $\delta\mathbb{S}$ of the slope region $\mathbb{S}$ and not in the interior $\mathbb{S} \setminus \delta\mathbb{S}$ [11, Lemma 2].

Contraction of the low contrast edges in the monotonic paths without eliminating the critical points, preserves the monotonicity of the path. A sequence of contractions may generate self-loops and multiple edges. In such cases, the slope region is difficult to analyze. The next section deals with the prototypes of the slope region in which the slope regions can be categorized.

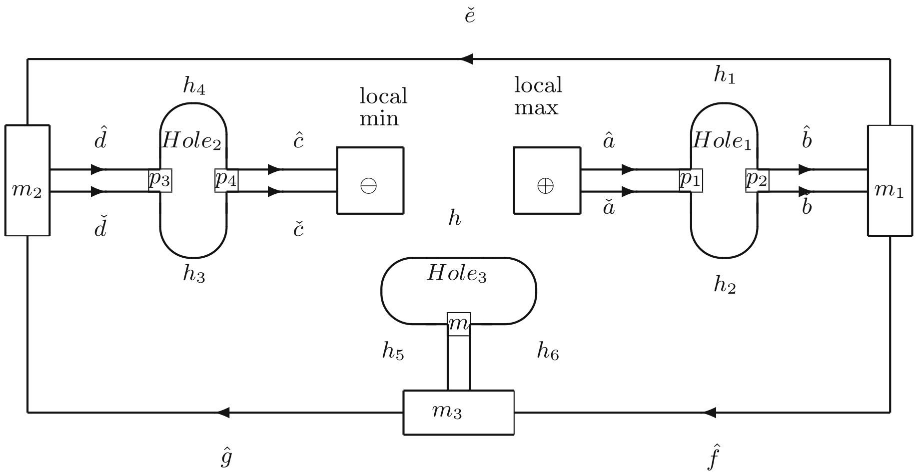
3 Two Prototypes of the Slope Regions

A prototype of a slope region is a graphical representation including all the possible components that a slope region may consist of. In other words, a slope region conforming to a prototype will obey the properties of the prototype. In this section, we explain the two prototypes of the slope region i.e. horizontal and inclined, by analyzing the similarities and the differences between them. Prototype 1 - Inclined slope region prototype is the extended version of the slope region prototype mentioned in [11]. The graphical and 2.5D representation of the prototype 1 and the prototype 0 are showed in Figs. 1 and 2 respectively. Existence and properties of holes in slope regions are described in Sect. 4.

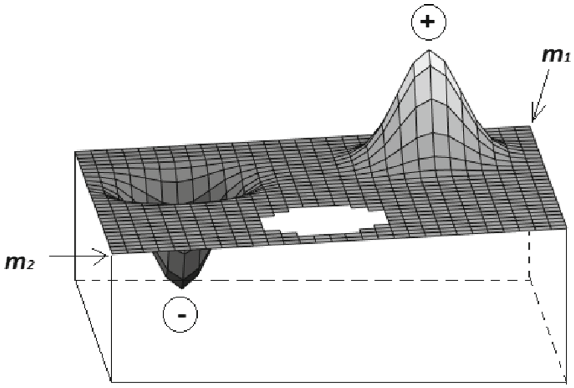
3.1 Components and Similarity Between the Two Prototypes

Figures 1(a) and 2(a) show all the components of prototype 1 - the inclined slope region and prototype 0 - the horizontal slope region respectively. Horizontal slope region prototype, in short is called as prototype 0 since it has zero inclination unlined the inclined slope region prototype. Both of them consists of at most two extrema (one local maximum $\oplus$ and one local minimum $\ominus$) geometrically inside the boundary of the slope region and are connected to the boundary via a monotonic path. All the elements such as $\oplus, \ominus$, holes are required to be connected to the boundary of the slope region to avoid generation of a disconnected graph. Therefore we have paths $(\oplus, m_1)$ and $(m_2, \ominus)$ connecting $\oplus$ and $\ominus$ to the boundary at vertex m_1 and m_2 respectively. The paths $(\oplus, m_1)$ and $(m_2, \ominus)$ must be monotonic to satisfy the Definition 7. A saddle point can only be present on the boundary of the slope region as stated in Remark 2. There might exist points (for example a local maximum or a local minimum disregarding $\oplus$ and $\ominus$) within the boundary which cannot be connected to either of the two extremum with a monotonic curve. Collections of all such points, geometrically inside the boundary of the slope region are classified as holes. Semantically, holes are not part of the slope region, since slope region is homeomorphic to the disk.

A hole in the slope region which do not lie on the monotonic path (for example: $Hole_3$ in Figs. 1 and 2) is surrounded by a self-loop and is connected to the boundary of the slope region by a level curve.



(a) Graphical representation of Prototype 1: inclined slope region



(b) Example of a mesh plot of prototype 1 without $Hole_1$ and $Hole_2$

Fig. 1. Prototype 1: inclined slope region.

A complete traversal of the *potentially folded boundary* of the inclined slope region is an alternating sequence of vertices and edges: $(\oplus, \hat{a}, p_1, h_1, p_2, \hat{b}, m_1, \breve{e}, m_2, \hat{d}, p_3, h_4, p_4, \hat{c}, \ominus, \check{c}, p_4, h_3, p_3, \check{d}, m_2, \breve{g}, m_3, h_5, m, h, m, h_6, m_3, \hat{f}, m_1, \check{b}, p_2, h_2, p_1, \check{a}, \oplus)$. Traversal of the folded boundary for horizontal slope region prototype can be worked out in similar fashion.

For a continuous case, all the gray values in the range $[g(\ominus), g(\oplus)]$ exist on the potentially folded boundary of the slope region. Any component, i.e., $\oplus$, $\ominus$ and holes may be excluded from the prototype to form a catalogue of all the possible slope regions conforming to the prototype 1 or prototype 0 respectively. An example of a mesh plot of prototype 1 and prototype 0 without $Hole_1$ and $Hole_2$ can be viewed in Fig. 1(a).

Orientation of Paths in Inclined Slope Region: All the paths are oriented following Definition 1 of edge orientation, i.e., from the higher gray-value vertex to the lower gray-value vertex. The two paths $\hat{a}$ and $\check{a}$ are on the same curve but are defined separately to complete the closed walk along the boundary of the slope region. $\hat{a}$ is the walk from $\oplus$ to $Hole_1$ following the orientation of the path from $\oplus$ to $Hole_1$. $\check{a}$ is an up-hill walk from $Hole_1$ to $\oplus$ in the direction opposite to the oriented path connecting $\oplus$ and $Hole_1$. Same applies to all the paths: $\hat{b}, \check{b}, \hat{c}, \check{c}, \hat{d}, \check{d}, h_5, h_6$.

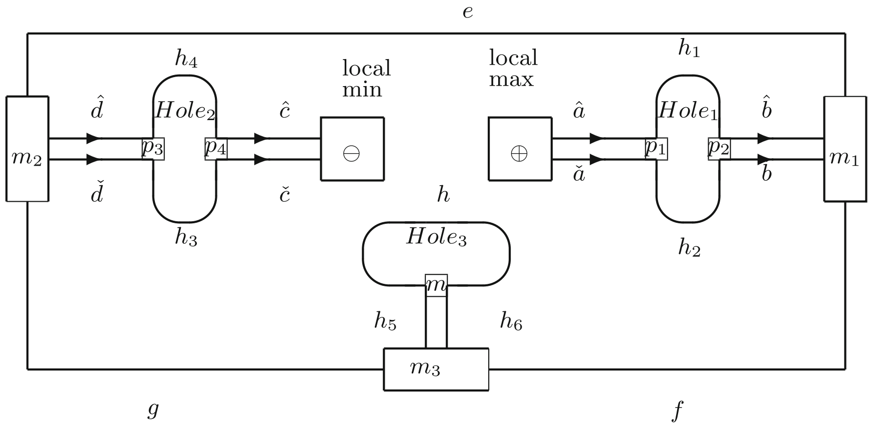
3.2 Difference Between the Two Prototypes

The only difference between the two prototypes is that, the outer boundary of Prototype 1 consists of exactly two monotonic paths $(m_1, \check{e}, m_2)$ and $(m_1, \hat{f}, m_3, \hat{g}, m_2)$. In contrast to the Prototype 1, the outer boundary of Prototype 0 is made up of level curve $(m_1, e, m_2, g, m_3, h_5, m, h, h_6, f)$ i.e. all the points on the boundary have same gray value. Consequently, it is possible that the extrema and holes are connected to only one vertex on the level curve, which would be the result of contraction of edges connecting (m_1, m_2) and (m_2, m_3) forming a self-loop. Hence the boundary in the self-loop slope region is non-oriented.

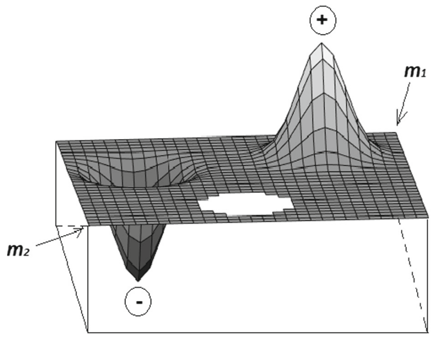
Note 1. The outer boundary of the Prototype 1 consists of exactly two monotonic paths while the outer boundary of the Prototype 0 is a single level curve.

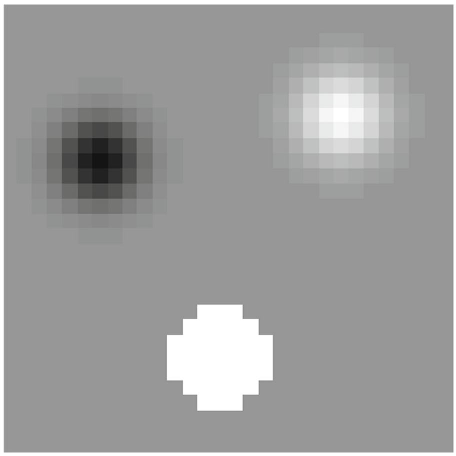
4 Holes in the Slope Region

Collections of all the points within the boundary of the slope region, which cannot be connected to the other points with a monotonic curve are classified as holes in the slope region and the points inside the hole do not belong to the slope region. Holes can be distinguished into two types depending on the connection of the hole with he boundary of the slope region. In Sect. 4.1 we describe the properties and condition for the hole connected to the boundary with a level curve and in Sect. 4.2 for the hole connected to the boundary with a monotonic path.



(a) Graphical representation of Prototype 0: horizontal slope region





(b) Mesh plot of prototype 0 (c) Gray scale image of prototype 0

Fig. 2. Prototype 0: horizontal slope region.

4.1 Hole Outside the Monotonic Path ($Hole_3$)

Collection of points which do not appear on the monotonic paths ($\oplus$, m_1) and ($m_2, \ominus$) and which cannot be connected to either $\oplus$ and $\ominus$ by a monotonic curve will be classified as a hole: $Hole_3$.

In prototype 1, as all the gray values in the range $[g(m_1), g(m_2)]$ exist on the outer boundary. Every point in one of the two monotonic paths of the boundary will be connected via an isoline (level-curve) to its counterpart point (with the same gray value) on the other monotonic path. If the boundary of the hole intersects multiple isolines, it will generate pair of points which can not be anymore connected by monotonic curves. This will invalidate the property of the slope region. Hence a single isoline will surround holes which are outside the monotonic path.

The hole must be connected to the boundary of the slope region by a level curve (cf., connection of vertices m and m_3 in Figs. 1(a) and 2(a)). In contradiction, if the hole was connected to the boundary with a path other than level curve, the pair of points on the either side of this path would not be connected by a monotonic curve. Similarly observations can be made for $Hole_3$ in prototype 0.

In case of multiple holes of same category, all the holes will be individually connected to the boundary of the slope region with a level curve. Since the hole is surrounded by a level curve, it must conform to prototype 0.

4.2 Holes on Monotonic Paths ($\oplus$, m_1) and ($m_2, \ominus$): ($Hole_1$ and $Hole_2$)

A hole on a monotonic path needs to intersect at least two distinct points to invalidate the monotonicity of the path. Referring to Figs. 1(a) and 2(a), $Hole_1$ intersects the monotonic path $(\oplus, m_1)$ at two distinct points p_1 and p_2.

Figure 3(a) and (b) shows the simplest example of existence of such hole which are formed by a local maximum and a local minimum (disregarding $\oplus$ and $\ominus$) respectively. In Fig. 3(a), the curve marked in yellow color will surround the hole with gray value $g(u)$. Similarly in Fig. 3(b), the curve marked in yellow color will surround the hole with gray value $g(l)$. In both the cases, the region between the range $[g(u), g(l)]$ (visible region between red and yellow curve) can be connected to the rest of the slope region with a monotonic curve, and hence will not be considered as a hole. Corresponding contour plots of Fig. 3(a) and (b) can be seen in Fig. 3(c) and (d) respectively where the boundary of the holes are marked in black. In prototype 1 and 0, referring to Figs. 1(a) and 2(a), the gray value of the path h_1 and h_2 connecting the two vertices p_1 and p_2 will be decided depending whether the hole is encapsulating a local maximum or a local minimum inside it. If the hole encapsulates a local maximum, the boundary of the hole equals to $g(u)$ and in case of minimum it equals to $g(l)$ as shown in Fig. 3. Same applies to the boundary of $Hole_2$ appearing on the monotonic path $m_2, \ominus$ in both the prototypes. Since the hole is surrounded by a level curve, it must conform to prototype 0.

4.3 Multiple Holes on Monotonic Paths

As we have already showed in the previous section that the boundary of the hole is a level curve, in Fig. 4 $g(p_1) = g(p_2)$ and $g(p_3) = g(p_4)$. In case of multiple holes appearing on the same monotonic path, the boundaries of different holes will be connected to each other either by a level curve or a monotonic path depending on the gray values of the level curve surrounding the holes. Referring to Fig. 4, if the gray values of the level curves surrounding the holes: $Hole_1$ and $Hole_2$ are the same, then the boundaries of the holes are connected by a level curve. Otherwise, they are connected by a monotonic path with orientation defined in Definition 1,

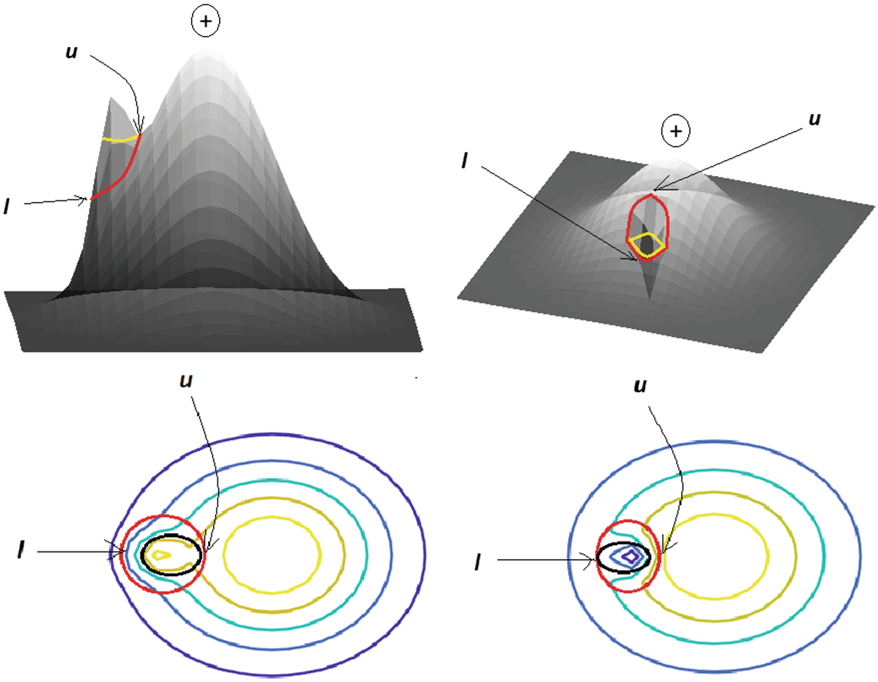


Fig. 3. Examples of mesh and contour plot of a hole on the monotonic path surrounding a local maximum (left) and a local minimum (right). (Color figure online)

i.e., from higher (p_2) to lower (p_3) gray value. The orientation will preserve the monotonicity of the path $(\oplus, m_1)$ and $(m_2, \ominus)$ in both the prototypes.

5 Partitioning of 2D Digital Image into Prototypes of Slope Region

Remark 3. A 2D image can be partitioned into the slope regions which can be categorized into one of the two prototypes: 1. inclined slope region and 2. horizontal slope region.

Proof. We already know that all the faces in the RAG G of a well-composed sampled surface are slope regions [2, Lemma 1]. The boundary of the slope region is composed by either a level curve or exactly two monotonic paths connecting the local maximum to the local minimum [11, Lemma 1]. The prototype of the *inclined slope region* (Sect. 3) corresponds to the slope region surrounded by exactly two monotonic paths. The prototype of *horizontal slope region* corresponds to the slope region surrounded by a level curve. Consecutively all the holes in both the enumerations follow the prototype 0 - *horizontal slope region*. The prototype satisfies the basic condition, that any pair of points inside the

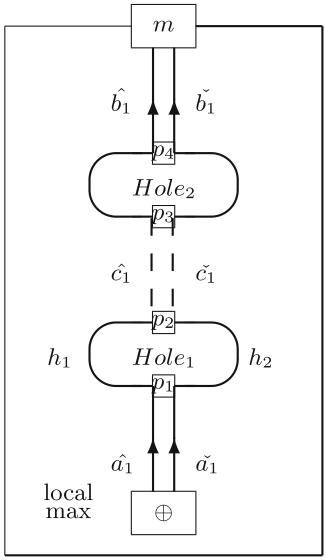


Fig. 4. Graphical representation of multiple holes on the monotonic path.

slope region can be connected by a monotonic curve. Thus we can span a 2D digital image with these two prototypes of slope regions. In other words, a 2D digital image can be partitioned into these two prototypes. □

6 Slope Regions Connected to Saddle Point

Lemma 1. *Presence of a saddle point guarantees existence of at least two slope regions.*

Proof. We prove Lemma 1 by contradiction. We already know that a saddle point cannot exist inside the slope region. A saddle point can only occur on the boundary of the slope region (Remark 2). A saddle point requires a minimum of 4 edges and at least 4 bit switches in the LBP coding (cf. Definition 4). When a saddle point is on the boundary of the inclined slope, it has at least two oriented edges or at most three oriented edges (if either of u or l in Fig. 1 are saddle points) connected to it. The saddle point on the boundary with two incident edges: 1. oriented inwards from local maximum towards to saddle point and 2. oriented outward from saddle point towards the local minimum. Let us assume that the other two edges incident on the saddle are inside the slope region. Referring to Fig. 5, the saddle point will require two additional edges which are connected to s_{max} and s_{min}. Also we know that for every point on the monotonic path $\check{a}$ connecting u and l, there exists a point on another monotonic path $(\hat{f}, \hat{c})$ connecting u and l, which can be connected by a level curve. Thus we cannot connect $\ominus$ and s_{min} with a monotonic curve. Similarly we also cannot connect s_{max} and $\oplus$ with a monotonic curve. This results in contradiction to our Definition 7 of slope region. We can prove the same for the saddle point (on u and l) with three incident edges. In case of horizontal slope region, the point m in Fig. 2 can be a saddle point on the boundary of the slope region the proof by contradiction holds good. Hence the presence of a saddle point guarantees existence of at least two slope regions, as one slope region is insufficient to satisfy the conditions of a saddle point.

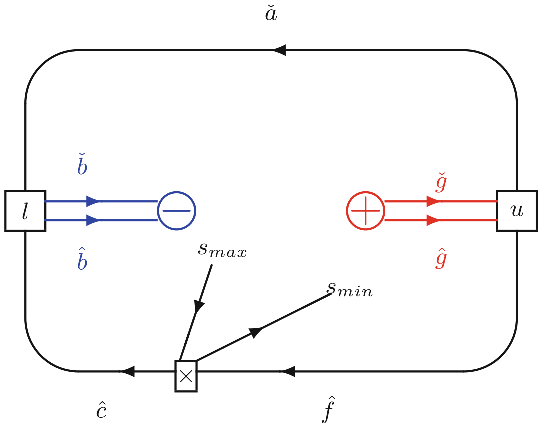


Fig. 5. Saddle on the boundary of a single slope region.

7 Conclusion

We introduced two prototypes of slope region: 1. inclined slope region and 2. horizontal slope region. We showed that a well composed 2D digital image can be partitioned into slope regions categorized into one of these two prototypes. We described the properties of the different types of holes which may appear inside a slope region and show that all holes follow prototype 0. We exploited the properties and connections incident to a saddle point on the boundary of the slope region. With the presence of a single saddle point, the property guarantees existence of at least two slope regions connected to the saddle point. The prototypes of the slope region along with the property of the saddle point introduced in this paper can form a grammar to generate digital images. We leave this topic and related questions for future research.

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