Technical Report

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Dual Contraction of Combinatorial Maps

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Abstract

This paper presents a new formalism for irregular pyramids based on combinatorial maps. The combinatorial map formalism allows us to encode a planar graph thanks to two permutations encoding the edges and the vertices of the graph.The combinatorial map formalism encode explicitly the orientation of the planar graph. This last property is useful to describe the partitions of an image which may be considered as a subset of the oriented plane $\mathbb{R}^2$. This new constraint allows us to design interesting properties for irregular pyramids. Finally the combinatorial formalism allows us to encode efficiently the graph transformations used in irregular pyramids.

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1 Introduction

The decomposition of an image into connected components, called the segmentation of an image, is necessary when we want to take decisions from an image or more generally when we want to analyze the different objects that compose this image. Unfortunately, this decomposition is not unique and the definition of a good segmentation often depends of the application. Moreover, the same application may need to have several levels of details in the decomposition of a same image. For example, a table in a living room may be defined as a unique connected component, or as five components describing its foots and its surface.

The multi-level representation of an image called pyramid [7, 11] allows us to define different levels of representation of the same object. This concept introduced by Pavlidis[7] allows one to manage the different representation levels of a partition by linking each connected component defined at one level with its decomposition in the next level. For example, the five connected components describing a table at one level should be unified into one component at an upper level. The top of a pyramid is usually composed of only one connected region describing the whole image while its base describes the lowest level of representation available on the image. For example, given a grey-scale image, the base of a pyramid can be composed of connected components having the same grey level. Another usual way to define the base of the pyramid consists to define each pixel of the input image as a basic region.

The first implementation of pyramids [12, 8, 14] shown in figure 1 use a regular tessellation of the image into a set of squares describing a balanced quadtree. Using such a representation the different regions are the nodes of a quadtree [15]. Thus a given region can only be represented by its lowest including square or its biggest included one. Jolion, Meer, and Montanvert [8, 9] have introduced the concept of irregular pyramids.

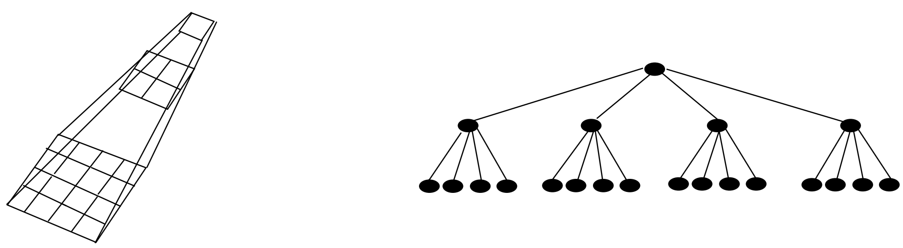


Figure 1: *A matrix pyramid and its associated quadtree*

The rest of the paper is organized as follows: In Section 2 and 3 we give some definitions and basic properties of graphs in the Combinatorial map formalism. In section 4 we demonstrate some intermediate properties which will be used in section 5. This last section includes the main results that will be used for irregular pyramids. Finally, in Section 6 and 7 we define respectively the Decimation Parameter and Contraction Kernel notions in terms of Combinatorial maps.

2 Combinatorial maps notions

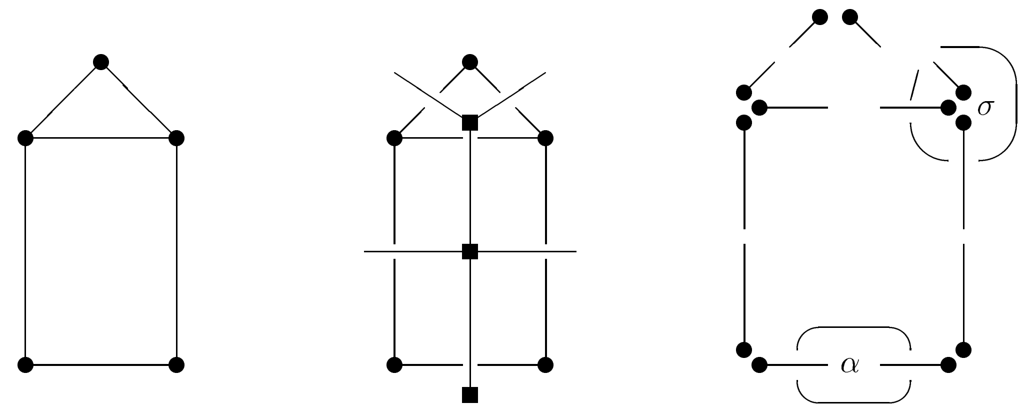
A Topological map [17, 16, 18] is a partition of a surface by a set of topological spaces called segments isomorph to the interval $I = [0,1]$ or to the circle S^1. The following definition of a Topological map has been given by Gareth [6]:

Definition 1 Topological map

Given a non empty set E of topological spaces (called edges*) each homeomorphic to the closed interval I=[0,1] or the circle* S^1*,together with a subset* $V \subset G = \cup_{e \in E} e$ *(V is called the set of vertices), such that if* δe *denotes* $e \cap V$ *and* $e^* = e - \delta e$*, then :*

1. *If e is homeomorphic to* S^1 *then* $|\delta e| = 1$ *(and e is called a loop), while if e is homeomorphic to I then* δe *consists of either one or both of the end points of e (and e is a free edge or a segment respectively)*
2. *For all distinct* e_1*,* e_2 *in E,* $e_1^* \cap e_2^* = \emptyset$*.*
3. *For any v in V, at most finely many e in E satisfy* $v \in \delta e$

If the topological map is embedded in the Euclidean space $\mathbb{R}^2$, it can be represented by a planar graph (see Figure 2-a). A planar graph, in its basic form does not include any orientation concept. It is only composed of a set of vertices, with for each vertex a set of incident edges. If we want to encode the orientation of the space, the planar map may be efficiently encoded with the **Combinatorial map** formalism.



(a) A plane graph (b) decomposed along dual edges (c) combinatorial map

Figure 2: From a plane graph to a combinatorial map

Figure 2 and 3 demonstrates the derivation of a combinatorial map from a plane graph. First edges are split where their dual edges cross (see Figure 2-b). That decomposes the graph into connected parts of half-edges that surround each vertex. These half edges are called *darts* and have their origin at the vertex they are attached to. The fact that two half-edges (darts) stem from the same edge is recorded in the **reverse permutation** α. A second permutation σ, called the **successor permutation**, defines the (local) arrangement of darts around a vertex. Counterclockwise ordering is assumed here (see Figure 3). More formally, we have:

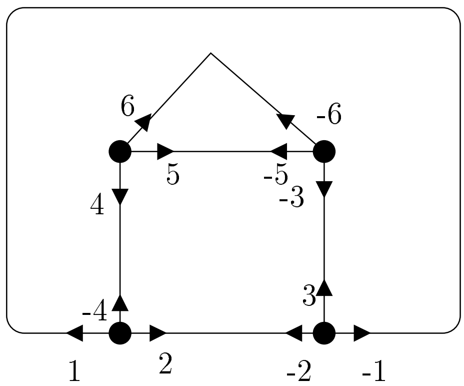
Definition 2 Combinatorial map

A combinatorial map G is the triplet $G = (\mathcal{D}, \sigma, \alpha)$, where $\mathcal{D}$ is a set called the set of darts and σ, α are two permutations defined on $\mathcal{D}$ such that α is an involution:

$$\forall d \in \mathcal{D} \quad \alpha^2(d) = d$$

If the darts are encoded by positive and negative integers, the permutation α can be implicitly encoded by $\alpha(d) = -d$ (see Figure 3). In the following, we will use alternatively both notations, the notation $\alpha(d) = -d$ will be often use for practical results linked to the implementation of our model. Indeed, if the permutation α is implicitly encoded, the combinatorial map may be implemented by a basic array of integers encoding the permutation σ.

Edmonds [19] and Gareth [6] shown that each Topological map (i.e. a partition of the space into segments) may be associated to a combinatorial map. The two following definitions comes from an article of Gareth [6] and will be used in the following.



$$\sigma = (1, 2, -4)(-2, -1, 3)(-3, -6, -5)(4, 5, 6)$$

Figure 3: *The permutation σ*

Definition 3 Group associated to a combinatorial map

Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$, the associated group $\mathcal{G}$ of G is the subgroup generated by σ and α within the symmetric group of all permutations on $\mathcal{D}$.

Definition 4 Morphism between combinatorial maps

Given two combinatorial maps $G_1 = (\mathcal{D}_1, \sigma_1, \alpha_1)$, $G_2 = (\mathcal{D}_2, \sigma_2, \alpha_2)$ and their associated subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$. A morphism $\phi : G_1 \to G_2$ is a pair of functions (χ, ψ), $\chi : \mathcal{G}_1 \to \mathcal{G}_2$ and $\psi : \mathcal{D}_1 \to \mathcal{D}_2$, where χ is a group homomorphism such that :

$$\begin{aligned} \chi(\alpha_1) &= \alpha_2 \\ \chi(\sigma_1) &= \sigma_2 \end{aligned}$$

and ϕ respect the orientation:

$$\forall d \in \mathcal{D}_1 \left\{ \begin{array}{rcl} \psi(\alpha_1(d)) & = & \alpha_2(\psi(d)) \\ \psi(\sigma_1(d)) & = & \sigma_2(\psi(d)) \end{array} \right. \tag{1}$$

If ψ is bijective ϕ will be called an isomorphism.

In other words, we will say that we have a morphism $\phi = (\chi, \psi)$, between a combinatorial map $G_1 = (\mathcal{D}_1, \sigma_1, \alpha_1)$ and another one $G_2 = (\mathcal{D}_2, \sigma_2, \alpha_2)$ if the function ψ from $\mathcal{D}_1$ to $\mathcal{D}_2$ respect the orientation defined on both maps. Thus if ψ verifies equation 1.

For example if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two sets of darts, and if π is a bijective application from $\mathcal{D}_1$ to $\mathcal{D}_2$, we can show easily that the two combinatorial maps $G_1 = (\mathcal{D}_1, \sigma, \alpha)$ and $G_2 = (\mathcal{D}_2, \pi \circ \sigma \circ \pi^{-1}, \pi \circ \alpha \circ \pi^{-1})$,where σ and α are defined on $\mathcal{D}_1$, are isomorph. Thus, the labeling of darts is not fundamental since given a combinatorial map G we can build other combinatorial maps based on a different set of labels while preserving the isomorphism with the combinatorial map G.

3 Graph notions in terms of combinatorial maps

A combinatorial map may be seen as a planar graph encoding explicitly the orientation of edges around a given vertex. Thus all graph definitions used in irregular pyramids [10] such as end vertices, self loops, or degrees may be retrieved easily (see definitions 5, 6 and 7). The symbols $\alpha^*(d)$ and $\sigma^*(d)$ stand, respectively, for the α and σ orbits of the dart d. More generally, if d is a dart and π a permutation we will denote the π-orbit of d by $\pi^*(d)$. The cardinal of this orbit will be denoted $|\pi^*(d)|$ (see, for example, definition 7).

In the following, we will often use the same notations for orbits, and sets whenever there are no ambiguities. For example, if d is a dart, and $\mathcal{D}$ a set of darts, we will respectively note $|\pi^*(d)|$ and $|\mathcal{D}|$ the cardinal of the orbit of d and the cardinal of $\mathcal{D}$ even if the two mathematical objects are different. In the same way we will often note: $\pi^*(d) \cap \mathcal{D}$ the set of darts belonging simultaneously to the orbit of d and $\mathcal{D}$. Finally, if $\mathcal{D}$ is a set of darts and π a permutation we have :

- $\pi(\mathcal{D}) = \{\pi(d) \in \mathcal{D} \mid d \in \mathcal{D}\}$ and

- $\pi^*(\mathcal{D}) = \{d \in \mathcal{D} \mid \exists d' \in \mathcal{D} \text{ and } d \in \pi^*(d')\}$

Definition 5 End vertices

Given a dart d, we call the end vertices of the edge $\alpha^*(d) = (d, -d)$ the orbits $\sigma^*(d)$ and $\sigma^*(-d)$.

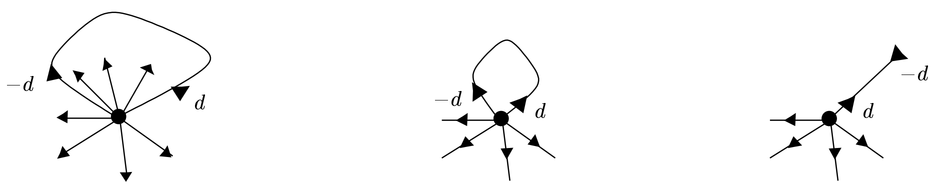
Definition 6 Self loop

An edge $\alpha^*(d)$ is called a self loop, iff: $-d \in \sigma^*(d)$

Definition 7 Degree

The vertex-degree of a dart d is equal to the cardinal of its σ-orbit $|\sigma^*(d)|$.

However, unlike planar graph the basic elements of a combinatorial map are not the edges but the half edges called darts. Thus instead of considering that an edge is shared by two vertices we will say that one dart belongs to one vertex. For example the vertex associated to dart 6 in Figure 3 is equal to $\sigma^*(6) = (6, 4, 5)$. This distinction allows us, for example, to define easily which of the extremity of an edge is pendant (see definitions 10 and 11). Moreover the orientation of darts around one vertex allows us to make a distinction between a self-loop (see definition 6) and a self-direct-loop (see definitions 8 and 9 and Figure 4).



(a) (b) (c)

Figure 4: *This figure illustrates the concepts of self-loop (a), self-direct-loop (b) and pendant dart (c)*

Definition 8 Dart self direct loop
A dart d is called a self direct loop, iff: $\sigma(d) = -d$. Note that if d or $-d$ is a self direct loop, the edge $\alpha^*(d)$ is a self loop. The reciprocity is false in general.

Definition 9 Edge self direct loop
An edge $\alpha^*(d)$ is called a self direct loop, iff one of its dart is a self direct loop.

Definition 10 Pendant dart
A dart d is called a pendant (or dangling) dart iff d is a fixed point of permutation σ, i.e. iff $\sigma(d) = d$. In this case the vertex $\sigma^*(d) = (d)$ is called a pendant vertex.

Definition 11 Pendant edge
An edge $\alpha^*(d)$ is called a pendant (or dangling) edge iff d or $-d$ are a pendant dart.

3.1 Paths, subgraph and connectedness

Each dart belonging to only one vertex it defines a natural orientation of the edge from $\sigma^*(d)$ to $\sigma^*(\alpha(d))$. Thus the usual notion of path in non-oriented planar graph is here replaced by oriented paths based on darts (see definition 12). The opposite of a path may be simply computed by taking the opposite $\alpha(d)$ of each dart d of the path (see proposition 1).

Definition 12 Path
A path in a combinatorial map is a sequence of darts $d_1, \ldots, d_n$ such that :

1. $\forall i \in \{1, \dots, n-1\}$ $\sigma^*(-d_i) \cap \{d_1, \dots, d_n\} = \{d_{i+1}\}$ *In other words, there is at most one edge which "arrives" on each vertex of the path.*

2. $d_n \notin \{d_1, \dots, d_{n-1}\}$ *and* $d_n \neq -d_{n-1}$

$\sigma^*(d_1)$ *and* $\sigma^*(-d_n)$ *are called the end vertices of the path. The number of darts of the path* P *is called its length and is denoted by* $|P|$.

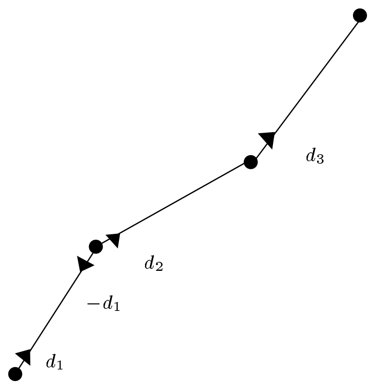


Figure 5: *A path of length 3*

Proposition 1 *Given a combinatorial map and a path* $P = d_1, \dots, d_n$ *in this map. The opposite path* $\alpha(P)$ *is defined by :*

$$\alpha(P) = -d_n, \dots, -d_1$$

In the same way, a circuit is defined as an oriented non-empty path leaving and reaching the same vertex (see definition 13).

Definition 13 Circuit

Given a combinatorial map G *and a path* $P = (d_1, \dots, d_n)$, *we will say that* P *is a circuit iff* $-d_n \in \sigma^*(d_1)$.

Proposition 2 *If a path* $P = (d_1, \dots, d_n)$ *is a circuit then :*

$$\sigma^*(-d_n) \cap \{d_1, \dots, d_n\} = \{d_1\}$$

Proof:

We have by definition of a circuit, $d_1 \in \sigma^*(-d_n)$. Let us suppose that we have :

$$\sigma^*(-d_n) \cap \{d_1, \dots, d_n\} = \{d_1, d_i\} \text{ with } i \geq 2$$

by definition $-d_{i-1}$ belong to the orbit of d_i. Thus we have :

$$\{d_i, d_1\} \subset \sigma^*(-d_{i-1}) \cap \{d_1, \dots, d_n\}$$

Which is in contradiction with the definition of a path. □

The connectedness of a combinatorial map may be defined in two equivalent ways. The definition 14 is based on paths and defines a connected map as a combinatorial map in which all vertices may be linked by a path. A less usual definition is given by Gareth et-Al [6] who have shown that a combinatorial map is connected iff its associated group is transitive (see definition 15 and proposition 3).

Definition 14 Connected Combinatorial Map
A combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ is said to be connected iff:

$$\forall d, d' \in \boldsymbol{\mathcal{D}} \quad \exists P = (d_1, \ldots, d_n) \quad | \quad P \text{ is a path and } d \in \sigma^*(d_1) \text{ and } d' \in \sigma^*(d_n)$$

Definition 15 Transitive group *If G is a group which operates on a set E, it is called transitive iff:*

$$\forall x, y \in E, \quad \exists g \in G \quad | \quad y = g(x)$$

Proposition 3 *A combinatorial map is connected iff its associated subgroup is transitive.*

3.2 Partition of a combinatorial map

Using combinatorial maps each vertex is implicitly defined by its set of darts. Thus a vertex partition of a combinatorial map may be defined by encoding each vertex by one of its dart (see definition 16). We say that a partition is minimal when each vertex is encoded by only one of its dart (see definition 17). Finally two partitions may form the same vertex partition with different set of darts, such partitions are called equivalent (see definition 18)

Definition 16 Partition
Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, $\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n \subset \boldsymbol{\mathcal{D}}$ is a vertex-partition of G iff:

1. $\forall i \in \{1, \ldots, n\} \quad \boldsymbol{\mathcal{D}}_i \neq \emptyset$

   All $\boldsymbol{\mathcal{D}}_i$ are non-empty.

2. $\forall d \in \boldsymbol{\mathcal{D}} \quad \exists i \in \{1, \ldots, n\}, \quad \exists d' \in \boldsymbol{\mathcal{D}}_i \quad | \quad d \in \sigma^*(d')$

   Each vertex may be retrieved thanks to a dart in one $\boldsymbol{\mathcal{D}}_i$.

3. $\forall i, k \in \{1, \ldots, n\}^2 \quad \sigma^*(\boldsymbol{\mathcal{D}}_i) \cap \sigma^*(\boldsymbol{\mathcal{D}}_k) = \emptyset$

   The set of darts of one vertex is included in only one $\boldsymbol{\mathcal{D}}_i$.

Note that we do not have $\bigcup_{i=1}^{n} \boldsymbol{\mathcal{D}}_i = \boldsymbol{\mathcal{D}}$. Condition 2, only requires that each vertex has at least one of its darts in one $\boldsymbol{\mathcal{D}}_i$.

Definition 17 Minimal partition

A partition $\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n$ of a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ is minimal iff:

$$\forall i \in \{1, \ldots, n\} \quad \forall d \in \boldsymbol{\mathcal{D}}_i \quad \sigma^*(d) \cap \boldsymbol{\mathcal{D}}_i = \{d\}$$

In other word, each vertex, is represented by only one dart.

Definition 18 Equivalent partition

Two partitions $\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n$ and $\boldsymbol{\mathcal{D}'}_1, \ldots, \boldsymbol{\mathcal{D}'}_n$ of a same combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ are said equivalent iff:

$$\forall i \in \{1, \ldots, n\} \quad \sigma^*(\boldsymbol{\mathcal{D}}_i) = \sigma^*(\boldsymbol{\mathcal{D}'}_i)$$

In other words, two partitions are equivalent, if each $\boldsymbol{\mathcal{D}}_i$ encodes the same vertices, with possibly different darts.

4 Properties of Parts of Combinatorial Maps

This section is mainly based on the function p (see Lemma 1) which allows us to restrict the permutation σ of a combinatorial map to a given set of darts. A first application of this function is the definition of a sub map (see proposition 4 and definition 19). Note that the notation $\sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}'}}(d)$ used in proposition 4 stands for $\sigma(p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}'}}(d))$. More generally, the sequence of mapping $f(g(d))$ will be denoted by $f \circ g(d)$ in the following for all dart d. Proposition 5 illustrates the validity of our submap definition by showing the transitivity of the submap relationship: Given three combinatorial maps , G_1, G_2 and G_3, if the combinatorial map G_2 is a submap of G_1 and if G_3 is a submap of G_2 then G_3 is also a submap of G_1.

Lemma 1 The Restriction Operator

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and $\boldsymbol{\mathcal{D}'} \subset \boldsymbol{\mathcal{D}}$ the application :

$$p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}'}} \left(\begin{array}{lll} \boldsymbol{\mathcal{D}'} & \rightarrow & \boldsymbol{\mathcal{D}} \\ d & \mapsto & \sigma^{n-1}(d) \textit{ with } n = Min\{p \in \mathbb{N}^* \mid \sigma^p(d) \in \boldsymbol{\mathcal{D}'}\} \end{array} \right.$$

is an injective function.

Proof:

$p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}'}}$ is defined since if $d' \in \boldsymbol{\mathcal{D}'}$, $n = |\sigma^*(d)|$ satisfies $\sigma^n(d) \in \boldsymbol{\mathcal{D}'}$. Let us suppose that we have two darts d_1 and d_2 in $\boldsymbol{\mathcal{D}}$' such that :$p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}'}}(d_1) = p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}'}}(d_2)$ then :

$$\exists n_1, n_2 \in \mathbb{N}^* \quad | \quad \sigma^{n_1}(d_1) = \sigma^{n_2}(d_2)$$

if $n_1 = n_2$ we have $d_1 = d_2$ since σ is bijective. Otherwise, let us suppose that $n_1 > n_2$, we have :

$$\sigma^{n_1 - n_2}(d_1) = d_2$$

Thus, $n_1 - n_2$ is smaller than n_1 and $\sigma^{n_1 - n_2}(d_1) \in \boldsymbol{\mathcal{D}'}$. This contradicts the requirement that n_1 is the minimum integer which verifies this property. $\square$

Proposition 4 *Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and $\mathcal{D}' \subset \mathcal{D}$ the application, $\sigma \circ p_{\mathcal{D},\mathcal{D}'}$ is a permutation.*

Proof:

We have to show that $\sigma \circ p_{\mathcal{D},\mathcal{D}'}$ is a bijective function on $\mathcal{D}'$.

The function $p_{\mathcal{D},\mathcal{D}'}$ is injective, σ is bijective, thus $\sigma \circ p_{\mathcal{D},\mathcal{D}'}$ is injective. Moreover, given a dart d in $\mathcal{D}'$ the set $\{n \in \mathbb{N}^* \quad | \quad \sigma^{-n}(d) \in \mathcal{D}'\}$ is non-empty since it contains all multiple of $|\sigma^*(d)|$. If n_1 is the minimum of this set, the dart $d' = \sigma^{-n_1}(d)$ verifies :

$$\sigma \circ p_{\mathcal{D},\mathcal{D}'}(d') = d$$

Thus $\sigma \circ p_{\mathcal{D},\mathcal{D}'}$ is bijective. Moreover, by definition, $\sigma \circ p_{\mathcal{D},\mathcal{D}'}$ applies from $\mathcal{D}'$ to $\mathcal{D}'$. Therefore, it is a permutation on $\mathcal{D}'$.

$\square$

Definition 19 Sub Combinatorial Map

Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$, $G' = (\mathcal{D}', \sigma', \alpha)$ is a submap of G iff:

1. $\mathcal{D}' \subset \mathcal{D}$
2. $\sigma' = \sigma \circ p_{\mathcal{D},\mathcal{D}'}$

This relation between G' and G will be denoted by $G' \subset G$.

Note that our definition of a sub combinatorial map is based on darts and not on vertices. Definition 20 provides a more usual definition. A vertex-induced sub combinatorial map is defined by a set of vertices and the set of edges linking these vertices.

Definition 20 Vertices Induced Sub combinatorial map

Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and a set of darts $\mathcal{D}' \subset \mathcal{D}$. The graph $G' = (\mathcal{D}'', \sigma \circ p_{\mathcal{D},\mathcal{D}''}, \alpha)$ with:

$$\mathcal{D}'' = \{d \in \mathcal{D} \quad | \quad \alpha^*(d) \subset \sigma^*(\mathcal{D}')\}$$

is a vertex induced sub-graph of G.

Lemma 2 *Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$, $\mathcal{D}' \subset \mathcal{D}$, the submap $G' = (\mathcal{D}', \sigma' = \sigma \circ p_{\mathcal{D},\mathcal{D}'}, \alpha)$ and $d \in \mathcal{D}'$ we have:*

$$\begin{array}{rcl} \sigma'^*(d) & = & (d) \text{ or} \\ \sigma'^*(d) & = & (d, d_1, \ldots, d_n) = (\sigma^0(d), \sigma^{p_1}(d), \ldots \sigma^{p_{n-1}}(d)) \text{ with} \\ & & 0 < p_1 < p_2 < \ldots < p_{n-1} < |\sigma^*(d)| \end{array}$$

Proof:

We have by definition of the restriction operator (see Lemma 1):

$$\sigma'(d) = \sigma^{p_1}(d) \text{ with } p_1 = Min\{n \in \mathbb{N}^* \quad | \quad \sigma^n(d) \in \boldsymbol{\mathcal{D}'}\}$$

Moreover: $\sigma^{|\sigma^*(d)|}(d) = d \in \boldsymbol{\mathcal{D}'}$, thus $p_1 \leq |\sigma^*(d)|$. If $p_1 = |\sigma^*(d)|$, we have $\sigma'^*(d) = (d)$.

Otherwise, let us suppose that the rank of $\sigma'^*(d)$ is equal to n with $n > 1$. Furthermore, we suppose that for all $i < n-1$:

$$\forall j \in \{1, \ldots, i\} \quad d_j = \sigma^{p_j}(d) \text{ with } 0 < p_{j-1} < p_j < |\sigma^*(d)|$$

we have:

$$\sigma'^{i+1}(d) = \sigma'(\sigma'^i(d)) = \sigma^{n_i+p_i}(d) = \sigma^{p_{i+1}}(d) \text{ with } n_i = Min\{n \in \mathbb{N}^* \quad | \quad \sigma^n(\sigma^{p_i}(d)) \in \boldsymbol{\mathcal{D}'}\}$$

We have, by hypothesis $p_i < |\sigma^*(d)|$ and $\sigma^{|\sigma^*(d)|-p_i}(\sigma^{p_i}(d)) = d \in \boldsymbol{\mathcal{D}'}$. Thus:

$$|\sigma^*(d)| - p_i \in \{n \in \mathbb{N}^* \quad | \quad \sigma^n(\sigma^{p_i}(d)) \in \boldsymbol{\mathcal{D}'}\}$$

Thus: $n_i \leq |\sigma^*(d)| - p_i$.

If $n_i = |\sigma^*(d)| - p_i$, we have $p_{i+1} = |\sigma^*(d)|$ and $\sigma'^{i+1}(d) = d$. Thus the rank of $\sigma'^*(d)$ is equal to i which is forbidden by hypothesis since $(i < n-1)$. Thus:

$$0 < n_i < |\sigma^*(d)| - p_i$$

Since $p_{i+1} = n_i + p_i$ we have:

$$0 < p_{i+1} < |\sigma^*(d)|$$

This recurrence will stop at step n where $\sigma'^n(d) = d = \sigma^{p_n}(d)$ and $p_n = |\sigma^*(d)|$. $\square$

Proposition 5 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and $\boldsymbol{\mathcal{D}}_2 \subset \boldsymbol{\mathcal{D}}_1 \subset \boldsymbol{\mathcal{D}}$. If G_1 is a sub combinatorial map deduced from G by $\boldsymbol{\mathcal{D}}_1$ and G_2 deduced from G_1 by $\boldsymbol{\mathcal{D}}_2$.*

$$G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha) \longrightarrow G_1 = (\boldsymbol{\mathcal{D}}_1, \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_1}, \alpha) \longrightarrow G_2 = (\boldsymbol{\mathcal{D}}_2, \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_1} \circ p_{\boldsymbol{\mathcal{D}}_1,\boldsymbol{\mathcal{D}}_2}, \alpha)$$

We have:

$$G_2 = (\boldsymbol{\mathcal{D}}_2, \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_2}, \alpha)$$

In other words, a sub-subcombinatorial map is a sub-combinatorial map.

Proof:

Let us note for simplicity $\sigma_1 = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_1}$.

Let us consider $d \in \boldsymbol{\mathcal{D}}_2$

$$\sigma_1 \circ p_{\boldsymbol{\mathcal{D}}_1,\boldsymbol{\mathcal{D}}_2}(d) = \sigma_1^{n_1}(d) \text{ with } n_1 = Min\{n \in \mathbb{N}^* \quad | \quad \sigma_1^n(d) \in \boldsymbol{\mathcal{D}}_2\}$$

We know, thanks to lemma 2 that:

$$\exists p_1 \in \{0, \ldots, |\sigma^*(d)| - 1\} \quad | \quad \sigma^{p_1}(d) = \sigma_1^{n_1}(d) \in \boldsymbol{\mathcal{D}}_2$$

Let us suppose that:

$$\exists p < p_1 \quad | \quad \sigma^p(d) \in \boldsymbol{\mathcal{D}}_2$$

We know, thanks to lemma 2 that:

$$\exists n < n_1 \quad | \quad \sigma_1^n(d) = \sigma^p(d) \in \boldsymbol{\mathcal{D}}_2$$

This is impossible since n_1 is the minimal power of σ_1 such that $\sigma_1^k(d) \in \boldsymbol{\mathcal{D}}_2$.

Therefore we cannot suppose the existence of p and we have:

$$\begin{array}{l} p_1 = Min\{p \in \mathbb{N}^* \quad | \quad \sigma^p(d) \in \boldsymbol{\mathcal{D}}_2\} \\ \Rightarrow \sigma^{p_1}(d) = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_2}(d) \end{array}$$

Finally:

$$\forall d \in \boldsymbol{\mathcal{D}}_2 \quad \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_2}(d) = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_1} \circ p_{\boldsymbol{\mathcal{D}}_1,\boldsymbol{\mathcal{D}}_2}(d)$$

□

The definition of a submap allows us to consider a component of a partition not only as a set of darts but also as a set of sub-maps. We may thus speak of the partition of a map into connected components (see definition 21). In the same way, we may define a cutset (see definition 22) which splits a map into a set of connected components.

Definition 21 Partition into Connected Components

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a partition $\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n$. This partition will be called a partition into connected components iff:

$$\forall i \in \{1, \ldots, n\} \left\{ \begin{array}{l} G_i = (\boldsymbol{\mathcal{D}}_i, \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}_i}, \alpha) \textit{ is connected} \\ \sigma^*(\boldsymbol{\mathcal{D}}_i) = \alpha(\sigma^*(\boldsymbol{\mathcal{D}}_i)) \end{array} \right.$$

The second equality means that there is no edge which connects $\boldsymbol{\mathcal{D}}_i$ to $\boldsymbol{\mathcal{D}}_j$.

Definition 22 Cutset

Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, $C \subset \boldsymbol{\mathcal{D}}$ will be called a cutset of G, iff:

1. *$G - C$ may be partitioned into connected components.*

2. *All subsets C' of $\boldsymbol{\mathcal{D}}$ which produce an equivalent partition include C.*

4.1 Dual Graphs

A face of a planar graph is defined by the set of edges which surround it. Using a combinatorial map, one dart per edge is sufficient to encode a face, since for each dart the involution α allows us to retrieve the other dart defining the edge. Moreover, the ordered sequence of darts around a vertex encoded by permutation σ induce an order in the sequence of faces encountered when turning around a face. This order is encoded thanks to the permutation $\varphi = \sigma \circ \alpha$ [5] (see definition 23).

Definition 23 Dual Combinatorial Map

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, the combinatorial map $\overline{G} = (\boldsymbol{\mathcal{D}}, \varphi, \alpha)$ is called the dual of G. The permutation φ is defined by:

$$\varphi = \sigma \circ \alpha$$

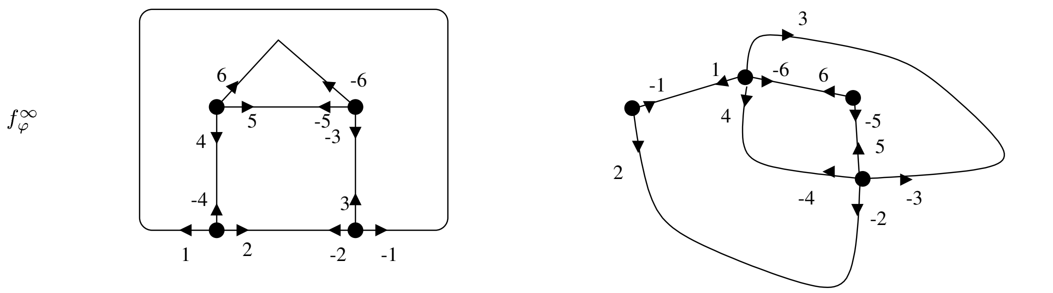
The orbits of φ encode the faces of G. Note that the function φ is a permutation, since it is the composition of two permutations on the same set.

Using a clockwise orientation for permutation σ all the faces of the combinatorial map except one are counter-clockwise oriented. The clockwise oriented face is called the infinite face (see definition 24). Using the correspondence between the topological and combinatorial maps, this face encodes the complementary in $\mathbb{R}^2$ of the union of the combinatorial map faces.

Definition 24 Infinite face

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, all the orbits of φ except one are clockwise oriented. This orbit encodes the infinite face of the graph and is denoted by f_φ^∞.

For example, the wall of the house in Figure 6 is encoded by the orbit: $\varphi^*(-4) = (-4, 5, -3, -2)$. This face can be read in the combinatorial map by turning counter-clockwise around the face or directly in the dual. In this last case the wall's face is represented by one vertex. Moreover, the darts of the $\varphi^*(-4)$ orbit link this vertex to the vertices associated to faces adjacent to the wall. The infinite face of this combinatorial map is defined by the orbit $\varphi^*(-1) = (-1, 2)$ which is clockwise oriented.



$\sigma = (1, 2, -4)(-2, -1, 3)(-3, -6, -5)(4, 5, 6)$ $\qquad$ $\varphi = (-1, 2)(-4, 5, -3, -2)(-5, 6)(-6, 4, 1, 3)$

(a) (b)

Figure 6: *A combinatorial map (a) and its dual (b)*

Note that the dual graph is obtained by composing the permutation σ with the permutation α which is bijective. Thus a combinatorial map and its dual contain the same information. This observation is confirmed by proposition 6 which shows that the original combinatorial map may be retrieved from its dual.

Proposition 6 *The Dual operation is idempotent.*

Proof:

If $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ is a combinatorial map we have:

$$\begin{array}{rcl} \overline{\overline{G}} & = & (\boldsymbol{\mathcal{D}}, \varphi \circ \alpha, \alpha) \\ & = & (\boldsymbol{\mathcal{D}}, \sigma, \alpha) \\ & = & G \end{array}$$

□

4.2 Dual Notions

We have seen in the previous section that the dual combinatorial map and the original one are deduced from each other by a very simple and bijective transformation. Therefore, we can expect that some properties true in the original combinatorial map remains true in the dual one. The proposition 7 and Corollary 1 show that the connectivity is preserved by the dual transformation.

Proposition 7 *G and $\overline{G}$ have the same associated subgroup.*

Proof:

The subgroup associated to G is generated by σ and α. The permutation α being an involution, the subgroup generated by $\sigma \circ \alpha$ and α is equal to G ($\sigma \circ \alpha \circ \alpha = \sigma$). □

Corollary 1 *If G is connected $\overline{G}$ is connected.*

Proof:

Both subgroups being identical, proposition 3 provides the result. □

In the same way, many particular configurations such as self-loop, or pendant edges remains particular configurations in the dual map. Indeed the proposition 8 shows that a pendant dart in the original map is mapped into a self-direct loop in the dual combinatorial map. The dual operation being idempotent a self-direct loop is mapped into a pendant dart in the dual. The proposition 9 (see Figure 7) generalizes these results to self-loops and bridges (see definition 25). All these dual notions are resumed in Table 1

Proposition 8 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a dart $d \in \boldsymbol{\mathcal{D}}$:*

- *If d is a pendant dart in G, $-d$ is a self direct loop in $\overline{G}$.*
- *If d is a self direct loop in G, $-d$ is a pendant dart in $\overline{G}$.*

Proof:

The dart $d \in \boldsymbol{\mathcal{D}}$ is a pendant dart in G iff $\sigma(d) = d$. Since $\sigma = \varphi \circ \alpha$ we have $\varphi(-d) = d$, thus $-d$ is a self-direct-loop of $\overline{G}$.

In the same way, if $\sigma(d) = -d$ we have $\varphi(-d) = -d$. Thus $-d$ is a pendant dart of $\overline{G}$. □

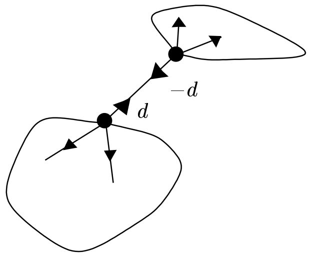


Figure 7: *A bridge between two connected components*

Definition 25 Bridge
Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and a dart $d \in \mathcal{D}$, the edge $\alpha^*(d)$ will be called a bridge iff :

$$-d \in \varphi^*(d)$$

Proposition 9 *Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$, its dual $\overline{G} = (\mathcal{D}, \varphi = \sigma \circ \alpha, \alpha)$, and a dart $d \in \mathcal{D}$ we have:*

1. *If d is a pendant dart, $\alpha^*(d)$ is a bridge. Thus a pendant edge is also a bridge.*
2. *The edge $\alpha^*(d)$ is a bridge in G iff it is a self loop in $\overline{G}$.*
3. *The edge $\alpha^*(d)$ is a pendant edge in G iff it is a self direct loop in $\overline{G}$.*

Proof:

First proposition: If d is a pendant dart, we have $\sigma(d) = d$, thus $\varphi(-d) = d$. Therefore the darts d and $-d$ belongs to the same orbits i.e. $-d \in \varphi^*(d)$.

Second proposition: This proposition may be immediately deduced from the definitions of bridge and self loop.

Third proposition: The edge $\alpha^*(d)$ is a pendant edge if one of its dart is a pendant dart. Let us suppose that $\sigma(d) = d$. Then we have $\varphi(-d) = d$, d is a dart-self-direct-loop in $\overline{G}$, thus $\alpha^*(d)$ is an edge-self-direct-loop in $\overline{G}$. The second implication may be shown in the same way.

□

5 Removal and Contraction operations

This section is devoted to the definition and the properties of the operations that will be used in irregular pyramids. Given a combinatorial map a first useful operation is the removal of an edge $\alpha^*(d)$. The resulting combinatorial map may be defined as a sub combinatorial map deduced from the original one by simply removing the darts d and

$(\boldsymbol{\mathcal{D}}, \sigma, \alpha)$	$(\boldsymbol{\mathcal{D}}, \varphi, \alpha)$
vertex $\sigma^*(d)$	orbit $(\varphi \circ \alpha)^*(d)$
orbit $(\sigma \circ \alpha)^*(d)$	face $\varphi^*(d)$
self-loop $-d \in \sigma^*(d)$	bridge $-d \in \varphi^*(d)$
bridge	self-loop
self-direct-loop	pendant dart
pendant dart	self-direct-loop

Table 1: Correspondence between particular configurations in the original and dual combinatorial maps

$\alpha(d)$ from its set of darts (see definition 26). Note that using definition 26, bridges are excluded from removal operations. This restriction allows us to preserve the number of connected components of the original combinatorial map.

Definition 26 Removal Operation
Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and $d \in \boldsymbol{\mathcal{D}}$. If $\alpha^*(d)$ is not a bridge, the combinatorial map $G' = G \setminus \alpha^*(d)$ is the submap defined by:

- $\boldsymbol{\mathcal{D}}' = \boldsymbol{\mathcal{D}} - \alpha^*(d)$ *and*
- $\sigma' = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}'}$.

This operation will be denoted $\boldsymbol{R_d}$.

The formal definition of the removal operation may be written in terms of modifications of permutation σ (see proposition 10). Note that self-direct loops are excluded from this proposition, in this case some simpler modifications of the combinatorial should be performed (see proposition 13).

Proposition 10 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a dart $d \in \boldsymbol{\mathcal{D}}$ which is neither a bridge nor a self-direct-loop, the submap $G \setminus \alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma', \alpha)$ is defined by:*

$$\left\{ \begin{array}{lcl} \forall d' \in \boldsymbol{\mathcal{D}} - \{\sigma^{-1}(d), \sigma^{-1}(-d)\} & & \sigma'(d') = \sigma(d') \\ \sigma'(\sigma^{-1}(d)) & = & \sigma(d) \\ \sigma'(\sigma^{-1}(-d)) & = & \sigma(-d) \end{array} \right.$$

Proof:
Given $d' \in \boldsymbol{\mathcal{D}} - \{\sigma^{-1}(d), \sigma^{-1}(-d)\}$ we have $\sigma(d') \notin \alpha^*(d)$ thus:

$$\sigma'(d) = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}-\alpha^*(d)}(d') = \sigma(d')$$

Since $\alpha^*(d)$ is neither a self direct loop nor a bridge, we have $\sigma^{-1}(d) \notin \alpha^*(d)$ and $\sigma(d) \neq -d$. Thus:

$$\sigma'(\sigma^{-1}(d)) = \sigma^2(\sigma^{-1}(d)) = \sigma(d)$$

In the same way:

$$\sigma'(\sigma^{-1}(-d)) = \sigma^2(\sigma^{-1}(-d)) = \sigma(-d)$$

□

The removal operation is defined only for one edge. If several removal operations have to be performed we have to check if the final combinatorial map depends or not of the order in which the removal operations are performed. The proposition 11 and the corollary 2 insure that the resulting combinatorial map is independent of the order of removal operations. Note that this result is a necessary condition for the design of an efficient parallel algorithm.

Proposition 11 Commutativity of Removal operations *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and two darts d_1 and d_2 in $\boldsymbol{\mathcal{D}}$, such that $d_1 \neq d_2$ and $G' = (\boldsymbol{\mathcal{D}} - \{d_1, d_2\}, \sigma \circ p_{\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}} - \{d_1, d_2\}}, \alpha)$ is connected, we have:*

$$\boldsymbol{R_{d_1}} \circ \boldsymbol{R_{d_2}}(G) = \boldsymbol{R_{d_2}} \circ \boldsymbol{R_{d_1}}(G)$$

Proof:

First, the removal of dart d_1 (resp. d_2) in the subgraph $G \setminus \alpha^*(d_2)$ (resp. $G \setminus \alpha^*(d_1)$) is defined, since G' is connected. Thus none of these darts will become a bridge.

The combinatorial map $\boldsymbol{R_{d_2}}(G)$ is a submap of G defined by $(\boldsymbol{\mathcal{D}} - \{d_2\}, \sigma \circ p_{\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}} - \{d_2\}}, \alpha)$. Moreover, the combinatorial map $\boldsymbol{R_{d_1}} \circ \boldsymbol{R_{d_2}}(G)$ is a sub-map of $\boldsymbol{R_{d_2}}(G)$ defined by:

$$\boldsymbol{R_{d_1}} \circ \boldsymbol{R_{d_2}}(G) = (\boldsymbol{\mathcal{D}} - \{d_1, d_2\}, \sigma \circ p_{\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}} - \{d_2\}} \circ p_{\boldsymbol{\mathcal{D}} - \{d_2\}, \boldsymbol{\mathcal{D}} - \{d_1, d_2\}}, \alpha)$$

Thanks to proposition 5 we have:

$$\boldsymbol{R_{d_1}} \circ \boldsymbol{R_{d_2}}(G) = (\boldsymbol{\mathcal{D}} - \{d_1, d_2\}, \sigma \circ p_{\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}} - \{d_1, d_2\}}, \alpha)$$

This last formula being symmetric in d_1 and d_2 we have:

$$\boldsymbol{R_{d_1}} \circ \boldsymbol{R_{d_2}}(G) = \boldsymbol{R_{d_2}} \circ \boldsymbol{R_{d_1}}(G)$$

□

Corollary 2 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a set of darts $d_1, \ldots, d_n$ such that:*

$$\begin{cases} \forall (i,j) \in \{1, \ldots, n\}^2, i \neq j \quad d_i \neq d_j \textit{ and} \\ G' = (\boldsymbol{\mathcal{D}} - \{d_1, \ldots d_n\}, \sigma \circ p_{\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{D}} - \{d_1, \ldots, d_n\}}, \alpha) \textit{ is connected} \end{cases}$$

For any permutation π defined on $d_1, \ldots, d_n$ we have:

$$\boldsymbol{R_{d_1}} \circ \ldots \circ \boldsymbol{R_{d_n}}(G) = \boldsymbol{R_{\pi(d_1)}} \circ \ldots \circ \boldsymbol{R_{\pi(d_n)}}(G)$$

In the following the removal of a set of darts $\boldsymbol{\mathcal{D}}$' will be denoted $\boldsymbol{R_{\mathcal{D}'}}$. The resulting combinatorial map will be alternatively denoted $\boldsymbol{R_{\mathcal{D}'}}(G)$ or $G \setminus \alpha^*(\boldsymbol{\mathcal{D}'})$.

Proof:

We can show easily from the hypothesis of this proposition that none of the darts $d_1, \ldots, d_n$ may becomes a bridge. Thus the removal operations are allowed in any order.

We can easily show, thanks to proposition 11, that this proposition is true for all transposition which permute two consecutive darts d_i, d_{i+1}. Since all permutations may be decomposed into a composition of such transposition, this property is true for all permutation. $\square$

Given a partition of an image, the merge of two regions may be considered in two different ways: First we can consider that the merge of the two regions is performed by removing one of their common boundaries. This operation is encoded in our combinatorial map formalism by the edge removal. Secondly, we can also consider that the merge of the two regions is performed by the identification of the two regions and the removal of one of their common boundaries. This dual point of view is encoded in our formalism by the contraction operation (see definition 28 and figure 9). Note that the identification operation may also be encoded thanks to definition 27 (see Figure 8).

Definition 27 Dart identification

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and one dart d, in $\boldsymbol{\mathcal{D}}$.
The combinatorial map $G' = (\boldsymbol{\mathcal{D}}, \sigma', \alpha)$ is said obtained from G by the vertex $d-$identification iff:

$$\begin{cases} \forall d \in \boldsymbol{\mathcal{D}} - \{d, \sigma^{-1}(-d)\} \quad \sigma'(d) = \sigma(d) \\ \sigma'(d) \quad = \quad -d \\ \sigma'(\sigma^{-1}(-d)) \quad = \quad \sigma(d) \end{cases}$$

In Figure 8 dart identification is illustrated by identifying dart 5 and dart -5 from Figure 3.

$\sigma = (1, 2, -4)(-2, -1, 3)(-3, -6, 6, 4, 5, -5)$ $\qquad$ $\sigma = (1, 2, -4)(-2, -1, 3)(-3, -6, -5, 5, 6, 4)$

Identification of dart 5 $\qquad$ Identification of dart -5

Figure 8: *Dart identification*

Definition 28 Contraction operation

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and one dart d, in $\boldsymbol{\mathcal{D}}$ which is not a self loop. The contraction of dart d creates the graph:

$$G' = G/\alpha^*(d) = \overline{\overline{G} \setminus \alpha^*(d)}$$

This operation will be denoted $\boldsymbol{C_d}$.

Note that this operation is well defined since d *is a self-loop in* G *iff it is a bridge in* $\overline{G}$.

Remark 1 *Note that, under the same hypothesis, we have:*

$$\overline{\overline{G}/\alpha^*(d)} = G \setminus \alpha^*(d)$$

Thus this two dual point of view on the merge of regions are performed by two dual operations on the combinatorial map and its dual. Thus many particular cases of one operation may be retrieved thanks to the particular cases the other. For example, since bridges are forbidden for removal operation the dual of a bride, i.e. a self-loop, is forbidden for contraction. In the same way the decomposition in two cases (see proposition 10 and proposition 13) used for the description of the removal operation in terms of modifications of permutation σ induces two cases for the equivalent description of contraction operation.(see propositions 12 and 13).

Proposition 12 *Given a combinatorial map* $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ *and a dart* $d \in \boldsymbol{\mathcal{D}}$ *which is neither a pendant edge nor a self loop. The graph* $G/\alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma', \alpha)$ *may be defined by:*

$$\left\{ \begin{array}{lll} \forall d' \in \boldsymbol{\mathcal{D}} - \sigma^{-1}(\alpha^*(d)) & \sigma'(d) = \sigma(d) & \\ \sigma'(\sigma^{-1}(d)) & = & \sigma(-d) \\ \sigma'(\sigma^{-1}(-d)) & = & \sigma(d) \end{array} \right.$$

Proof:

If d is is neither a pendant edge nor a self loop in G, it is neither a self-direct-loop nor a bridge in $\overline{G} = (\boldsymbol{\mathcal{D}}, \varphi, \alpha)$ (see proposition 8). Thus $\overline{G} \setminus \alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \varphi', \alpha)$ may be defined by:

$$\left\{ \begin{array}{lll} \forall d \in \boldsymbol{\mathcal{D}} - \{\varphi^{-1}(d), \varphi^{-1}(-d)\} & \varphi'(d) = \varphi(d) & \\ \varphi'(\varphi^{-1}(d)) & = & \varphi(d) \\ \varphi'(\varphi^{-1}(-d)) & = & \varphi(-d) \end{array} \right.$$

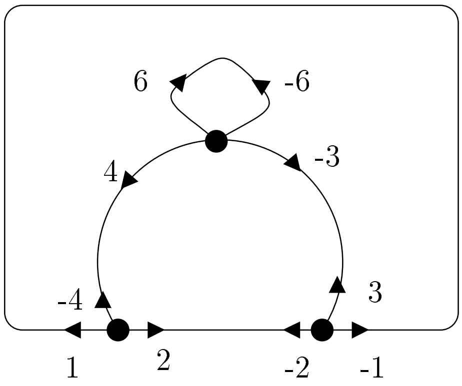
Since $G/\alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma', \alpha) = \overline{\overline{G} \setminus \alpha^*(d)} = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \varphi' \circ \alpha, \alpha)$. We have $\sigma' = \varphi' \circ \alpha$. Moreover, we have $\varphi^{-1} = \alpha \circ \sigma^{-1}$. A simple substitution of this two last equality in the equations defining $\overline{G} \setminus \alpha^*(d)$ provides the result. $\Box$

Tutte [18] also defines contractions and removal operations in the combinatorial map formalism. However, Tutte defines these two operations on PreMaps which are not necessarily oriented, and are thus too general for our purpose. Moreover, using Tutte's approach, the proposition 10 and 12 are taken as definitions. In this case the definitions 26 and 28 must become theorems.

Proposition 13 Study of some particular cases *Given a combinatorial map* $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ *and a dart* $d \in \boldsymbol{\mathcal{D}}$.

Removal operation: *Let us consider* $G \setminus \alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma', \alpha)$

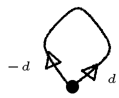
1. *If* $\sigma(d) = -d$ *and* $\sigma(-d) = d$ *(isolated)*



$$\sigma = (1,2,-4)(-2,-1,3)(-3,-6,6,4)$$

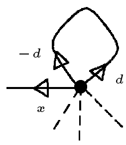
Figure 9: *Contraction of edge $\alpha^*(5)$*

$$G \setminus \alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma, \alpha)$$



2. *If $\sigma(d) = -d$ and $\sigma(-d) = x \notin \alpha^*(d)$ (self direct loop)*

$$\sigma'(\sigma^{-1}(d)) = \sigma(-d) = x$$

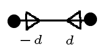


Contraction operation: *Let us consider $G/\alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma', \alpha)$*

All the following properties are deduced from the removal operation in $\overline{G}$

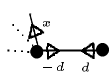
1. *If $\sigma(d) = d$ and $\sigma(-d) = -d$ (pendant edge)*
   $\iff \varphi(-d) = d$ *and* $\varphi(d) = -d$.

$$\begin{array}{rcl} \overline{G} \setminus \alpha^*(d) & = & (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma \circ \alpha, \alpha) \\ \Rightarrow \quad G/\alpha^*(d) & = & (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma, \alpha) \end{array}$$



2. *If $\sigma(d) = d$ and $\sigma(-d) = x \notin \alpha^*(d)$ (pendant edge)*
   $\iff \varphi(-d) = d$ *and* $\varphi(d) = x \notin \alpha^*(d)$.

$$\begin{array}{rcccl} \varphi'(\varphi^{-1}(-d)) & = & \varphi(d) & = & x \\ \Rightarrow \quad \sigma'(\sigma^{-1}(-d)) & = & \sigma(-d) & = & x \end{array}$$



Proof:

All the equations relative to the removal operation may easily be deduced from the definition. The equations relative to the contraction are deduced from the one describing the removal operation. □

Moreover, some properties true for one operation may be easily deduced for the other thanks to the dual operation. For example, the commutativity of contraction operations (see proposition 14) is easily deduced from the same property for the removal operation (see proposition 11). Inversely, the connectivity of a combinatorial map obtained after a sequence of removal operation is deduced from the same property shown for the contraction operation (see proposition 15 and corollary 3).

Proposition 14 Commutativity of Contractions

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a set of darts $d_1, \ldots, d_n$ such that:

$$\begin{cases} \forall (i,j) \in \{1, \ldots, n\}^2, i \neq j \quad d_i \neq d_j \\ \boldsymbol{C_{d_1}} \circ \ldots \circ \boldsymbol{C_{d_n}}(G) \text{ is connected} \end{cases}$$

For any permutation π defined on $d_1, \ldots, d_n$ we have:

$$\boldsymbol{C_{d_1}} \circ \ldots \circ \boldsymbol{C_{d_n}}(G) = \boldsymbol{C_{\pi(d_1)}} \circ \ldots \circ \boldsymbol{C_{\pi(d_n)}}(G)$$

In the following, the contraction of a set of darts $\boldsymbol{\mathcal{D}'}$ will be denoted $\boldsymbol{C_{\mathcal{D}'}}$

Proof:

If $\boldsymbol{C_{d_1}} \circ \ldots \circ \boldsymbol{C_{d_n}}(G)$ is connected, its dual, $\boldsymbol{R_{d_1}} \circ \ldots \circ \boldsymbol{R_{d_n}}(\overline{G})$ is also connected. Thus the hypothesis of corollary 2 are verified. Moreover,

$$\begin{cases} \overline{\boldsymbol{C_{d_1}} \circ \ldots \circ \boldsymbol{C_{d_n}}(G)} & = & \boldsymbol{R_{d_1}} \circ \ldots \circ \boldsymbol{R_{d_n}}(\overline{G}) \\ & = & \boldsymbol{R_{\pi(d_1)}} \circ \ldots \circ \boldsymbol{R_{\pi(d_n)}}(\overline{G}) \text{ (see corollary 2)} \\ & = & \overline{\boldsymbol{C_{\pi(d_1)}} \circ \ldots \circ \boldsymbol{C_{\pi(d_n)}}(G)} \end{cases}$$

Thus:

$$\boldsymbol{C_{d_1}} \circ \ldots \circ \boldsymbol{C_{d_n}}(G) = \boldsymbol{C_{\pi(d_1)}} \circ \ldots \circ \boldsymbol{C_{\pi(d_n)}}(G)$$

□

Lemma 3 Link between the orbits of G and $G/\alpha^*(d)$

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ a dart $d \in \boldsymbol{\mathcal{D}}$ which is neither a self loop nor a pendant edge and $G' = G/\alpha^*(d) = (\boldsymbol{\mathcal{D}} - \alpha^*(d), \sigma', \alpha)$ we have:

$$\forall d' \in \boldsymbol{\mathcal{D}} - \sigma^*(\alpha^*(d)) \quad \sigma'^*(d') = \sigma^*(d')$$

Moreover, if $|\sigma^*(d)| > 3$ we have:

$$\sigma'^*(\sigma(d)) = (\sigma(d), \ldots, \sigma'(\sigma(d))^{|\sigma(d)|-2}, \sigma(-d), \ldots, \sigma'(\sigma(-d))^{|\sigma(-d)|-2})$$

Thus, in this last case:

$$\forall d' \in \sigma^*(\alpha^*(d)) - \alpha^*(d) \quad d' \in \sigma'^*(\sigma(d))$$

Proof:

Note that we are in the hypothesis of proposition 12. The equations describing the contraction in this case may thus be applied.

First equality: The first equality is an equality between orbits, we have thus to show that the two permutations σ and σ' are equal on $\sigma^*(d')$ and thus on $\sigma'^*(d')$. We have:

$$\begin{array}{ll} & \sigma^{-1}(\alpha^*(d)) \subset \sigma^*(\alpha^*(d)) \\ \Rightarrow & \boldsymbol{\mathcal{D}} - \sigma^*(\alpha^*(d)) \subset \boldsymbol{\mathcal{D}} - \sigma^{-1}(\alpha^*(d)) \\ \Rightarrow & \forall d' \in \boldsymbol{\mathcal{D}} - \sigma^*(\alpha^*(d)) \quad \sigma'(d') = \sigma(d') \end{array}$$

Since $d' \in \boldsymbol{\mathcal{D}} - \sigma^*(\alpha^*(d))$, we have $\sigma^*(d') \subset \boldsymbol{\mathcal{D}} - \sigma^*(\alpha^*(d))$. Thus $\sigma^*(d') = \sigma'^*(d')$ and both permutations are equal on $\sigma^*(d')$.

Second equality: Let us consider:

$$\begin{array}{rcl} n_1 & = & |\sigma^*(d)| - 1 \\ n_2 & = & |\sigma^*(-d)| - 1 \\ n & = & n_1 + n_2 \end{array}$$

And the serie $d_0, \ldots, d_n$ such that: $d_i = \sigma'^i(\sigma(d))$.

We have:
$d_0 = \sigma'^0(\sigma(d)) = \sigma^0(\sigma(d)) = \sigma(d) \neq \sigma^{-1}(d)$.

Thus:
$d_1 = \sigma'(d_0) = \sigma(d_0) = \sigma^2(d) \neq \sigma^{-1}(d)$ since $(|\sigma^*(d)| > 3)$

Let us suppose that:
$d_j = \sigma^j(d) \neq \sigma^{-1}(d)$ for j in $\{1, \ldots, i\}$ for a given i in $\{2, \ldots, n_1 - 2\}$.

Since $d_i \neq \sigma^{-1}(d)$ we have

$$d_{i+1} = \sigma'(d_i) = \sigma(d_i) = \sigma^{i+1}(d)$$

Moreover, $i + 1 < n_1 - 1$, thus $d_{i+1} \neq \sigma^{-1}(d)$. We have:

$$\begin{array}{rcl} d_{n_1} & = & \sigma'(d_{n_1-1}) \\ & = & \sigma'(\sigma^{n_1-1}(d)) \\ & = & \sigma'(\sigma^{-1}(d)) \\ & = & \sigma(-d) \end{array}$$

In the same way, We can show by recurrence that:

$$\forall k \in \{n_1 + 1, n - 1\} \quad d_k = \sigma^{k-n_1}(-d)$$

We have thus:

$$\begin{array}{rcl} d_n & = & \sigma'(d_{n-1}) \\ & = & \sigma'(\sigma^{n_2-1}(-d)) \\ & = & \sigma'(\sigma^{-1}(-d)) \\ & = & \sigma(d) \end{array}$$

Thus $d_n = \sigma'^n(\sigma(d)) = \sigma(d)$ and n is the lowest integer which realizes this equality. The orbits of $\sigma(d)$ is thus equal to $(d_1, \ldots, d_n)$.

□

Proposition 15 Connectivity of $G/\alpha^*(d)$
Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and a dart $d \in \mathcal{D}$ which is not a self loop, if G is connected $G/\alpha^*(d)$ is connected.

Proof:
Let us consider two darts b_1 and b_2 in $\mathcal{D} - \alpha^*(d)$. Since G is connected, we have a path $P = d_1, \ldots, d_n$ in G such that:

$$\begin{array}{l} b_1 \in \sigma^*(d_1) \\ b_2 \in \sigma^*(-d_n) \end{array}$$

Let us consider three case:

case a) $:\forall i \in \{1, \ldots, n\} d_i \notin \sigma^*(\alpha^*(d))$

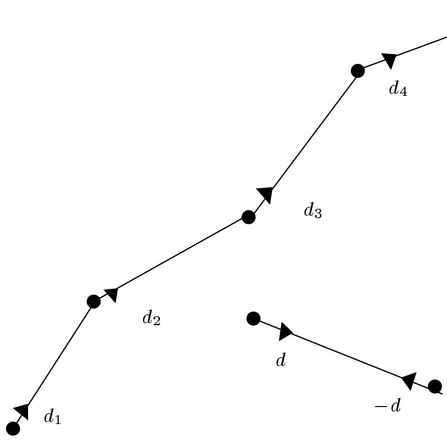


Figure 10: *Case a*

In this case, we have for all i:

$$\sigma'^*(d_i) = \sigma^*(d_i)$$

The orbits of the darts composing the path remains unchanged in G', and we can easily show that P remains a valid path in G'.

case b) $\exists! i \in \{1, \ldots, n\} \quad / \quad d_i \in \sigma^*(\alpha^*(d))$

Let us suppose that:$d_i \in \sigma^*(d)$ (the same demonstration hold for $d_i \in \sigma^*(-d)$)

We have $-d_{i-1} \in \sigma^*(d_i) = \sigma^*(d)$.

- If $\alpha^*(d)$ is a pendant edge. It may be easily show, that in this case $i \in \{2, \ldots, n-1\}$ and $d_i \neq d$. Thus, $-d$ is the pendant dart of $\alpha^*(d)$. Using proposition 13 we have:

$$\sigma'^*(-d_{i-1}) = \sigma^*(-d_{i-1}) - \{d\}$$

Thus:

$$\sigma'^*(-d_{i-1}) \cap \{d_1, \ldots, d_n\} = \sigma^*(-d_{i-1}) \cap \{d_1, \ldots, d_n\} = \{d_i\}$$

P remains a valid path in G'.

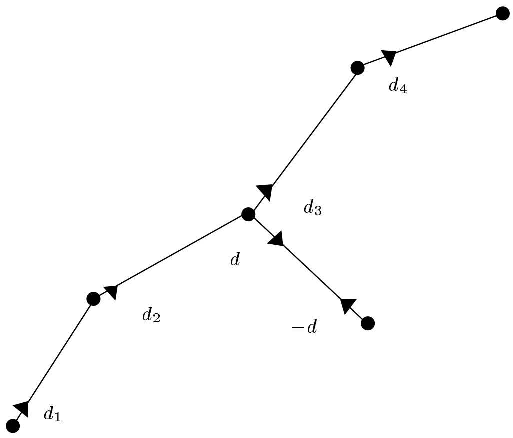


Figure 11: *Case b*

- If $\alpha^*(d)$ is not a pendant edge
  Since $\sigma^*(d)$ is included in $\sigma'^*(\sigma(d))$ (see Lemma 3), we have:

  $$\sigma'^*(d_i) = \sigma'^*(-d_{i-1}) = \sigma'^*(\sigma(d))$$

  The dart d_i being unique in P, we have for all k in $\{1, \ldots, n\}$ $d_k \notin \sigma^*(-d)$. Thus

  $$\{d_1, \ldots, d_n\} \cap \sigma^*(-d) = \emptyset$$

  . Otherwise, we know, thanks to Lemma 3 that $\sigma'^*(d)$ is composed of darts belonging to $\sigma^*(d)$ and $\sigma^*(-d)$. Thus:

  $$\{d_i\} \subset \sigma'^*(-d_{i-1}) \cap \{d_1, \ldots, d_n\} \subset (\sigma^*(d) \cup \sigma^*(-d)) \cap \{d_1, \ldots, d_n\} = \{d_i\}$$

  Thus $\sigma'^*(-d_{i-1}) \cap \{d_1, \ldots, d_n\} = \{d_i\}$. The other orbits being unchanged for all other darts composing the path P, P remains a valid path in G'.

case c) $\exists!(i,j) \in \{1, \ldots, n\}^2 i < j \quad | \quad \{d_i, d_j\} \subset \sigma^*(\alpha^*(d))$

Let us suppose that:

$$d_i \in \sigma^*(d)$$
$$d_j \in \sigma^*(-d)$$

In this case $\alpha^*(d)$ cannot be a pendant edge, and we can easily show thanks to Lemma 3 that:

$$\{d_1, \ldots, d_n\} \cap \sigma'^*(\sigma(d)) = \{d_1, \ldots, d_n\} \cap \sigma'^*(-d_{i-1}) = \{d_i, d_j\}$$

If we consider the path $P' = d_1, \ldots, d_{i-1}, d_j, \ldots, d_n$ we have:

$$\{d_1, \ldots, d_{i-1}, d_j, \ldots, d_n\} \cap \sigma'^*(\sigma(d)) = \{d_j\}$$

Moreover:$-d_{i-1} \in \sigma^*(d_i) = \sigma^*(d)$ thus, $\sigma'^*(-d_{i-1}) = \sigma'^*(\sigma(d))$. All the σ-orbits of darts $-d_k, d_k \in P'$ being unchanged, P' is a valid path in G'.

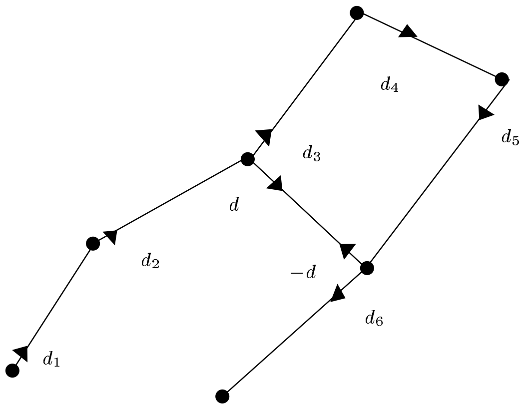


Figure 12: *Case c*

Other cases: If we have more than 3 darts of P in $\sigma^*(\alpha^*(d)) = \sigma^*(d) \cup \sigma^*(-d)$ at least 2 belong to the same σ-orbits. This is forbidden by the definition of a path.

$\square$

Corollary 3 Connectivity of $G \setminus \alpha^*(d)$
Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and a dart $d \in \mathcal{D}$ which is not a bridge, if G is connected $G \setminus \alpha^*(d)$ is connected.

Proof:
If d is not a bridge in G, it is not a self-loop in $\overline{G}$. Thus the contraction operation is allowed in the dual and $\overline{G}/\alpha^*(d)$ is connected. Thus, its dual is connected (see Corollary 1). Moreover, we have:

$$\overline{\overline{G}/\alpha^*(d)} = G \setminus \alpha^*(d)$$

Thus, $G \setminus \alpha^*(d)$ is connected. $\square$

6 Decimation parameters

In order to perform more than one contraction simultaneously or to avoid violating the precondition of contraction (e.g. creating a self-loop!) we have to impose a constraint on the set of darts to be contracted:

Definition 29 independent vertex set
Given a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$, a set of darts $\mathcal{D}' \subset \mathcal{D}$ will be called an independent vertex set iff:

$$\alpha(\sigma^*(\mathcal{D}')) \cap \sigma^*(\mathcal{D}') = \emptyset$$

The set $\mathcal{D}'$ will be called a maximum independent vertex set iff:

$$\forall d \in \mathcal{D} - \sigma^*(\mathcal{D}') \quad \exists d' \in \sigma^*(d) \quad | \quad -d' \in \sigma^*(\mathcal{D}')$$

All vertices are defined by one dart of $\boldsymbol{\mathcal{D}'}$ or are linked to one vertex defined by a dart in $\boldsymbol{\mathcal{D}'}$.

Definition 30 Decimation parameter
Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a set of darts $\boldsymbol{\mathcal{D}'} \subset \boldsymbol{\mathcal{D}}$ will be called a decimation parameter of G iff it is an independent vertex set and:

$$\forall d \in \boldsymbol{\mathcal{D}} - \sigma^*(\boldsymbol{\mathcal{D}'}) \quad \exists! d' \in \boldsymbol{\mathcal{D}'} \quad | \quad - d' \in \sigma^*(d)$$

Definition 31 Connecting Path *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a decimation parameter $\boldsymbol{\mathcal{D}}$' of G and two darts b_1 and b_2 in $\sigma^*(\boldsymbol{\mathcal{D}'})$, $CP(b_1, b_2)$ will be called a connecting path iff it is a path and if it verifies one of the following conditions:*

1. $CP(b_1, b_2) = d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}'})$
2. $CP(b_1, b_2) = d_1 d_2$ *with:*
$$|\{d_1, d_2\} \cap \alpha^*(\boldsymbol{\mathcal{D}'})| = 1$$
3. $CP(b_1, b_2) = d_1 d_2 d_3$ *with:*
$$|\{d_1, d_2, d_3\} \cap \alpha^*(\boldsymbol{\mathcal{D}'})| = 2$$

Lemma 4 *Given a combinatorial map without pendant edges $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a decimation parameter $\boldsymbol{\mathcal{D}'}$ and a dart $d \in \boldsymbol{\mathcal{D}}$, at least one of the two darts d, $\varphi(d)$ belongs to $\boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{D}'}$.*

Proof:
Let us suppose that both darts d and $\varphi(d)$ belong to $\boldsymbol{\mathcal{D}'}$. Then we have:

- $d \in \sigma^*(d) \subset \sigma^*(\boldsymbol{\mathcal{D}'})$ and
- $d \in \alpha(\sigma^*(\sigma(\alpha(d))) = \alpha(\sigma^*(\varphi(d))) \subset \alpha(\sigma^*(\boldsymbol{\mathcal{D}'}))$.

This is in contradiction with the definition of an independent vertex set. □

Proposition 16 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a decimation parameter $\boldsymbol{\mathcal{D}}$' of G any dart of $\boldsymbol{\mathcal{D}}$ which is not a pendant dart belonging to $\alpha^*(\boldsymbol{\mathcal{D}'})$ belongs at least to one connecting path.*

Proof:
Given a dart d in $\boldsymbol{\mathcal{D}}$, we have to find a path which contains it. Let us decompose this demonstration into three cases:

If $d \in \boldsymbol{\mathcal{D}'}$ We know thanks to Lemma 4 that $\varphi(d) \in \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{D}'}$. Moreover, if d is not a pendant dart, $\alpha(\varphi(d))$ cannot belong to $\boldsymbol{\mathcal{D}'}$ without violating the definition of a decimation parameter. Thus $P = d, \varphi(d)$ is a path, $d \in \boldsymbol{\mathcal{D}'}$ and $\varphi(d) \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}'})$. Thus, P is a connecting path.

If $d \in \alpha(\boldsymbol{\mathcal{D}'})$ If d is not a pendant dart, we have $\varphi^{-1}(d) \neq d$. Therefore:

- $\varphi^{-1}(d)$ cannot belong to $\boldsymbol{\mathcal{D}}'$ without violating the definition of a decimation parameter.
- $\varphi^{-1}(d)$ cannot belong to $\alpha(\boldsymbol{\mathcal{D}}')$ without violating the definition of an independent vertex set.

Thus $\varphi^{-1}(d)$ belongs to $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$. Moreover, by definition it exists a dart d' such that: $-d' \in \sigma^*(\varphi^{-1}(d))$. The series $P = d', \varphi^{-1}(d), d$ is a path and $|P \cap \alpha^*(\boldsymbol{\mathcal{D}}')| = 2$. The path P is thus a connecting path.

If $d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ Let us decompose this last case in three sub cases.

1. If $\sigma^*(d) \cap \boldsymbol{\mathcal{D}}' \neq \emptyset$ and $\sigma^*(\alpha(d)) \cap \boldsymbol{\mathcal{D}}' \neq \emptyset$. The series $P = d$ is a connecting path by definition.
2. If $\sigma^*(d) \cap \boldsymbol{\mathcal{D}}' = \emptyset$ and $\sigma^*(\alpha(d)) \cap \boldsymbol{\mathcal{D}}' \neq \emptyset$
   Then d belongs to $\boldsymbol{\mathcal{D}} - \sigma^*(\boldsymbol{\mathcal{D}}')$. Therefore, it exists a dart $d' \in \boldsymbol{\mathcal{D}}'$ such that $\alpha(d') \in \sigma^*(d)$. Then we have: $d' \in \boldsymbol{\mathcal{D}}'$, $d \notin \alpha^*(\boldsymbol{\mathcal{D}}')$, thus $P = d', d$ is a connecting path.
3. If $\sigma^*(d) \cap \boldsymbol{\mathcal{D}}' = \emptyset$ and $\sigma^*(\alpha(d)) \cap \boldsymbol{\mathcal{D}}' = \emptyset$
   We can find, by definition of a decimation parameter, two darts d' and d'' such that:
   - $\alpha(d') \in \alpha(\boldsymbol{\mathcal{D}}') \cap \sigma^*(d)$ and
   - $\alpha(d'') \in \alpha(\boldsymbol{\mathcal{D}}') \cap \sigma^*(\alpha(d))$

   The path $P = d', d, d''$ is thus a connecting path.

□

Proposition 17 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a decimation parameter $\boldsymbol{\mathcal{D}}$' of G, we have one and only one dart in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ for each connecting path.*

The connecting path which contain a dart d in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ will be denoted $CP(d)$.

Proof:

The existence of a connecting path for each dart in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ is given by proposition 16. The uniqueness is trivial from the definition. □

Note that the pendant darts which belong to the decimation parameter are excluded from proposition 16 since they cannot be part of a connecting path. If we contract all the darts of the decimation parameters, the pendant darts will be simply removed (see proposition 13). The definition of decimation parameters being devoted to the contraction operation these darts may be removed from the combinatorial map without loss of information.

Definition 32 Map without pendant edges

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a decimation parameter $\boldsymbol{\mathcal{D}}'$, the submap without pendant edges will be defined by $G' = (\boldsymbol{\mathcal{D}}'', \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}''}, \alpha)$ where:

$$\boldsymbol{\mathcal{D}}'' = \boldsymbol{\mathcal{D}} - \alpha^*(\{d \in \alpha^*(\boldsymbol{\mathcal{D}}') \mid \sigma^*(-d) = (-d)\})$$

The submap without pendant edges is simply the same map in which we have removed all pendant edges with one dart in $\boldsymbol{\mathcal{D}}$'. In order to simplify the notations, in the following, we will assume that this operation has been performed and thus, that $\alpha^*(\boldsymbol{\mathcal{D}}')$ does not have pendant edges.

Definition 33 Reversal of connecting paths
Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a decimation parameter $\boldsymbol{\mathcal{D}}$' of G, we define the involution α_C which associates to each connecting path containing one dart d of $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ the connecting path which contain $-d$. We have thus:

$$\forall d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}') \quad \alpha_C(CP(d)) = CP(\alpha(d))$$

Definition 34 Representative dart
Given a combinatorial map without pendant edges $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a decimation parameter $\boldsymbol{\mathcal{D}}'$, the representative of a dart d denoted $Rep(d)$ is equal to:

- *d if $d \notin \boldsymbol{\mathcal{D}}'$ and*
- *$\varphi(d)$ elsewhere.*

Proposition 18 *Given a combinatorial map without pendant edges $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a decimation parameter $\boldsymbol{\mathcal{D}}'$, the application:*

$$\sigma_C \left(\begin{array}{ccc} \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}') & \rightarrow & \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}') \\ d & \mapsto & \left(\begin{array}{l} Rep(\sigma(d)) \text{ if } -\sigma(d) \notin \boldsymbol{\mathcal{D}}' \\ Rep(\varphi(\sigma(d)) \text{ elsewhere} \end{array} \right. \end{array} \right.$$

is a bijective function.

Proof:
We have by definition of the representative dart (see definition 34), $\sigma_C(d) \notin \boldsymbol{\mathcal{D}}'$ for all d in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$. Let us suppose that $\sigma_C(d)$ belongs to $\alpha(\boldsymbol{\mathcal{D}}')$ for some d in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$. Then:

- If $-\sigma(d) \notin \boldsymbol{\mathcal{D}}'$

  Then $\sigma_C(d) = Rep(\sigma(d)) \in \alpha(\boldsymbol{\mathcal{D}}')$.

  We cannot suppose that $Rep(\sigma(d)) = \sigma(d)$, since in this case we have $\sigma_C(d) = \sigma(d) \in \alpha(\boldsymbol{\mathcal{D}}')$ which is in contradiction with the hypothesis.

  Thus $Rep(\sigma(d)) = \varphi(\sigma(d)) \in \alpha(\boldsymbol{\mathcal{D}}')$ and $\sigma(d) \in \boldsymbol{\mathcal{D}}'$. In this case, $\varphi(\sigma(d)) = \sigma(-\sigma(d))$ and $-\sigma(d)$ belong to the same σ-orbit and have their opposite in $\boldsymbol{\mathcal{D}}$'. This is forbidden by the definition of a decimation parameter.

- If $-\sigma(d) \in \boldsymbol{\mathcal{D}}'$

  Then $\sigma_C(d) = Rep(\varphi(\sigma(d)) \in \alpha(\boldsymbol{\mathcal{D}}')$

  If $Rep(\varphi(\sigma(d)) = \varphi(\sigma(d)) \in \alpha(\boldsymbol{\mathcal{D}}')$, then since $-\sigma(d)$ belongs to $\boldsymbol{\mathcal{D}}'$, we have $\sigma^*(-\sigma(d)) \subset \sigma^*(\boldsymbol{\mathcal{D}}')$. Moreover, $\varphi(\sigma(d)) = \sigma(-\sigma(d)) \in \sigma^*(-\sigma(d))$. Thus:

$$\varphi(\sigma(d)) \in \sigma^*(\boldsymbol{\mathcal{D}}') \cap \alpha(\boldsymbol{\mathcal{D}}')$$

This is forbidden by definition of an independent vertex set.

Let us suppose that $Rep(\varphi(\sigma(d))) = \varphi^2(\sigma(d)) \in \alpha(\boldsymbol{\mathcal{D}}')$, In this case, the darts $-\varphi(\sigma(d))$ and $\varphi^2(\sigma(d))$ belong to the same σ-orbit and have their opposite in $\boldsymbol{\mathcal{D}}'$. Once again this is forbidden by the definition of a decimation parameter.

Thus we have for all d in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ $\sigma_C(d) \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$. Let us show that σ_C is bijective.

Let us suppose that we have two darts d_1 and d_2 in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ such that $\sigma_C(d_1) = \sigma_C(d_2)$. Then, the different cases involved by this last equality may be decomposed in two "big" cases (The cases which are not mentioned in the following demonstration may be simply retrieved by exchanging d_1 and d_2).

1. If $-\sigma(d_1) \notin \boldsymbol{\mathcal{D}}'$ and $-\sigma(d_2) \notin \boldsymbol{\mathcal{D}}'$ Then we have:

   (a) $\sigma(d_1) = \sigma(d_2)$ or $\varphi(\sigma(d_1)) = \varphi(\sigma(d_2))$ or

   (b) $\sigma(d_1) = \varphi(\sigma(d_2))$

   The first case is trivial since σ and φ are bijective. The second case arise only when $\sigma(d_2) \in \boldsymbol{\mathcal{D}}'$. Moreover we have in this case $d_1 = -\sigma(d_2)$, thus d_1 belongs to $\alpha(\boldsymbol{\mathcal{D}}')$ which is forbidden by hypothesis.

2. If $-\sigma(d_1) \notin \boldsymbol{\mathcal{D}}'$ and $-\sigma(d_2) \in \boldsymbol{\mathcal{D}}'$ In this case we have:

   (a) $\sigma(d_1) = \varphi(\sigma(d_2))$ or $\varphi(d_1) = \varphi(\sigma(d_2))$ or

   (b) $\varphi(d_1) = \varphi^2(\sigma(d_2))$ or $\sigma(d_1) = \varphi^2(\sigma(d_2))$

   The first case involve that d_1 is equal to $\sigma(d_2)$ or $-\sigma(d_2)$ which is impossible since d_1 belongs to $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ and $-\sigma(d_2)$ to $\boldsymbol{\mathcal{D}}'$. In the same way, the second case involves that d_1 is equal to $\varphi(\sigma(d_2))$ or $-\varphi(\sigma(d_2))$. This is impossible since, this case arise only when $\varphi(\sigma(d_2))$ belongs to $\boldsymbol{\mathcal{D}}$' and that d_1 belongs to $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$.

The function σ_C is injective and applies from a set to itself. This function is thus bijective and defines a permutation on $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ $\square$

Remark 2 *Since we have one and only one dart of $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ for each connecting path, the permutation σ_C may be seen has a permutation on the set of connecting paths thanks to the following formula:*

$$\forall d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}') \quad \sigma_C(CP(d)) = CP(\sigma_C(d))$$

Definition 35 Connecting path map

Given a combinatorial map without pendant edges $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a decimation parameter $\boldsymbol{\mathcal{D}}'$, the set of connecting path $\boldsymbol{\mathcal{D}}_C$, may be defined by:

$$\boldsymbol{\mathcal{D}}_C = \{CP(d), d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')\}$$

The map of connecting paths associated to the decimation parameter $G\boldsymbol{\mathcal{D}}$' is defined by:

$$G\boldsymbol{\mathcal{D}}' = (\boldsymbol{\mathcal{D}}_C, \sigma_C, \alpha_C)$$

(see definition 33 and proposition 18).

Definition 36 Structure Preserving Contraction

Given a combinatorial map without pendant edges $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a decimation parameter $\boldsymbol{\mathcal{D}'}$. A combinatorial map $G' = (\boldsymbol{\mathcal{D}''}, \sigma', \alpha')$ will be called a structure preserving contraction according to $\boldsymbol{\mathcal{D}}$' iff it is isomorph (see definition 4) to the graph of connecting paths $G\boldsymbol{\mathcal{D}}$' associated to $\boldsymbol{\mathcal{D}}$'.

6.1 Link between connecting paths and contraction

Lemma 5 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a decimation parameter $\boldsymbol{\mathcal{D}}$', and the two successor permutations σ_C and σ' of respectively, the connecting path map $G\boldsymbol{\mathcal{D}'}$ and the contracted map $G' = G \setminus \alpha^*(\boldsymbol{\mathcal{D}'}) = (\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}'}), \sigma', \alpha')$ we have:*

$$\forall d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}) \quad \sigma_C(d) = \sigma'(d)$$

Proof:

Let us decompose demonstration in the four cases describing the different values that may be taken by $\sigma_C(d)$.

1. If $\alpha^*(\sigma(d)) \cap \boldsymbol{\mathcal{D}'} = \emptyset$ then we have

   $\sigma_C(d) = \sigma(d)$.

   In this case $\sigma'(d) = \varphi'(-d) = \varphi(-d) = \sigma(d)$ since $(\varphi(-d) \notin \alpha^*(\boldsymbol{\mathcal{D}'}))$.

2. If $-\sigma(d)) \notin \boldsymbol{\mathcal{D}'}$ and $\sigma(d) \in \boldsymbol{\mathcal{D}'}$ then we have

   $\sigma_C(d) = \varphi(\sigma(d)) = \varphi^2(-d) \notin \alpha^*(\boldsymbol{\mathcal{D}'})$ by definition of the permutation σ_C.

   Thus $\sigma'(d) = \varphi'(-d) = \varphi^2(-d)$. Indeed, $\varphi(-d) = \sigma(d)$ belong to $\boldsymbol{\mathcal{D}}$'.

3. If $-\sigma(d)) \in \boldsymbol{\mathcal{D}'}$ and $\varphi(\sigma(d)) \notin \boldsymbol{\mathcal{D}'}$ then,

   $\sigma_C(d) = \varphi(\sigma(d)) = \varphi^2(-d)$

   $\sigma'(d) = \varphi'(-d) = \varphi^2(-d)$ $(\varphi(-d) = \sigma(d) \in \alpha(\boldsymbol{\mathcal{D}'}))$

4. Finally, if $-\sigma(d)) \in \boldsymbol{\mathcal{D}'}$ and $\varphi(\sigma(d)) \in \boldsymbol{\mathcal{D}'}$ then,

   $\sigma_C(d) = \varphi^2(\sigma(d)) = \varphi^3(-d) \notin \alpha^*(\boldsymbol{\mathcal{D}'})$

   Then we have:
   $$\begin{cases} \varphi(-d) & = & \sigma(d) & \in & \alpha(\boldsymbol{\mathcal{D}'}) \text{ and} \\ \varphi^2(-d) & = & \varphi(\sigma(d)) & \in & \boldsymbol{\mathcal{D}'} \end{cases}$$

   Thus: $\sigma'(d) = \varphi(-d) = \varphi^3(-d)$

□

Theorem 1 *Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a decimation parameter $\boldsymbol{\mathcal{D}}$'. The contracted map $G' = G \setminus \alpha^*(\boldsymbol{\mathcal{D}'})$ is a structure preserving contraction*

Proof:

Let us denote respectively, $G\boldsymbol{\mathcal{D}}'$ and G' the connecting path map and the contracted one. We have:

$$\begin{array}{lcl} G\boldsymbol{\mathcal{D}}' & = & (\boldsymbol{\mathcal{D}}_C, \sigma_C, \alpha_C) \\ G' & = & (\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}'), \sigma', \alpha) \end{array}$$

Now, let us consider the application $\phi = (\chi, \psi)$ from $G\boldsymbol{\mathcal{D}}'$ to G' such that:

$$\chi : \begin{array}{lcl} \sigma_C & \mapsto & \sigma' \\ \alpha_C & \mapsto & \alpha \end{array}$$

$$\psi \left(\begin{array}{lcl} \boldsymbol{\mathcal{D}}_C & \rightarrow & \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}') \\ CP(d) & \mapsto & d \end{array} \right.$$

The function ψ is well defined and bijective since each connecting path have one and only one dart in $\boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}')$ (see proposition 17).

Let us show that ϕ is a morphism, thus that:

$$\forall CP(d) \in \boldsymbol{\mathcal{D}}_C \left\{ \begin{array}{lcl} \psi(\alpha_C(CP(d))) & = & \alpha(d) \\ \psi(\sigma_C(CP(d))) & = & \sigma'(d) \end{array} \right.$$

The first equality is trivial since, $\alpha_C(CP(d)) = CP(\alpha(d))$ (see definition 33). Thus:

$$\psi(\alpha_C(CP(d))) = \psi(CP(\alpha(d))) = \alpha(d)$$

The second equality may be easily deduced from Lemma 5. Indeed, we have

$$\forall d \in \boldsymbol{\mathcal{D}} - \alpha^*(\boldsymbol{\mathcal{D}}') \quad \sigma_C(d) = \sigma'(d)$$

Thus:

$$\psi(\sigma_C(CP(d))) = \psi(CP(\sigma_C(d))) = \sigma_C(d) = \sigma'(d)$$

□

7 Contraction Kernel

We will provide in this section a definition of a tree and a forest. These definitions will be used to define a contraction Kernel and the **connecting serie map** deduced from it. Finally we will show that the connecting serie map is isomorph to a given contracted map.

7.1 Tree and forest

As mentioned in section 5 a sequence of merge in a partition may be encoded by a sequence of contractions of the combinatorial map encoding the partition. Since the contraction operation is forbidden for self-loops the set of darts involve in such a sequence of contractions will not contain a circuit (see definition 13). Thus the set of edges involve in such a contraction may be encoded by a Tree (see definition 37).

Definition 37 Map tree
Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a set $\boldsymbol{\mathcal{D}}$' will be called a subtree of G iff $\alpha^*(\boldsymbol{\mathcal{D}}') = \boldsymbol{\mathcal{D}}'$ and the submap:

$$G_T = (\boldsymbol{\mathcal{D}}', \sigma' = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}'}, \alpha)$$

has only one φ'-orbit.

More generally, if we contract a set of vertices into a given set of surviving vertices, the set of darts involve in such contractions may be encoded by a forest (see definition 38).

Definition 38 Spanning Forest
Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, the set $F = (\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n)$ of trees will be called a spanning forest of G iff $\{\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n\}$ is a vertex partition of G.

7.2 Contraction Kernel

In the following we will focus on connected combinatorial map. If the combinatorial map is not connected the following definitions and propositions may be applied on each component of the combinatorial map. Moreover, the vertices of a combinatorial map being implicitly define by the darts which belong to their orbits, we must require that at least on edge survive. In this last case the resulting graph is reduced to one vertex with a self loop. The two previous restriction have been used in definition 39 to define a contraction kernel and the set of surviving darts.

Definition 39 Contraction Kernel
Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, the forest $F = (\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n)$ will be called a contraction kernel iff:

$$\mathcal{SD} = \boldsymbol{\mathcal{D}} - \bigcup_{i=1}^{n} \boldsymbol{\mathcal{D}}_i \neq \emptyset$$

The set $\mathcal{SD}$ is called the set of surviving darts.

Lemma 6 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a contraction kernel $F = (\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n)$ we have the following property:*

$$\forall i \in \{1, \ldots, n\} \quad \sigma^*(\boldsymbol{\mathcal{D}}_i) \cap \mathcal{SD} \neq \emptyset$$

Proof:
Let us consider a dart $d \in \boldsymbol{\mathcal{D}}_i$ and a dart $d' \in \mathcal{SD}$. The combinatorial map G being connected we have a path $P = d_1, \ldots, d_n$ from $\sigma^*(d)$ to $\sigma^*(d')$. Now let us consider d_i such that:

$$i = Max\{j \in \{1, \ldots, n\} \mid \forall k \in \{1, \ldots, j\} d_k \in \sigma^*(\boldsymbol{\mathcal{D}}_i)\}$$

Note that the index i is at least equal to 1 since $d_1 \in \sigma^*(d) \subset \sigma^*(\boldsymbol{\mathcal{D}}_i)$.

- Let us suppose that $i < n$

  Using item 3 of definition 16 we have $\sigma^*(d_i) \cap \bigcup_{j=1, j\neq i}^n \boldsymbol{\mathcal{D}}_j = \emptyset$. Thus $d_i \in \boldsymbol{\mathcal{D}}_i$ or $d_i \in \mathcal{SD}$. If d_i belongs to $\boldsymbol{\mathcal{D}}_i$, the tree $\boldsymbol{\mathcal{D}}_i$ being symmetric we have $\alpha(d_i) \in \boldsymbol{\mathcal{D}}_i$. Thus $d_{i+1} \in \sigma^*(\alpha(d_i)) \subset \sigma^*(\boldsymbol{\mathcal{D}}_i)$ which is in contradiction with the definition of d_i, thus we have $d_i \in \mathcal{SD}$. Since $d_i \in \sigma^*(\boldsymbol{\mathcal{D}}_i)$ the lemma is demonstrated.

- If $i = n$ Then we can show easily, with the same kind of demonstration, that $d_n \in \mathcal{SD} \cap \sigma^*(\boldsymbol{\mathcal{D}}_i)$ or $d_n \in \boldsymbol{\mathcal{D}}_i$. In this last case we have $\alpha(d_n) \in \boldsymbol{\mathcal{D}}_i$ and $d' \in \sigma^*(\boldsymbol{\mathcal{D}}_i) \cap \mathcal{SD}$.

□

Proposition 19 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a contraction kernel $F = (\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n)$ we have the following property:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \varphi^*(d) \cap \mathcal{SD} \neq \emptyset$$

Proof:

The proposition is trivial if $d \in \mathcal{SD}$. Let us suppose that d belongs to a given $\boldsymbol{\mathcal{D}}_i \in F$. The tree $\boldsymbol{\mathcal{D}}_i$ being symmetric we have $\alpha(d) \in \boldsymbol{\mathcal{D}}_i$. Using item 3 of definition 16 we have $\sigma^*(\alpha(d)) \cap \bigcup_{j=1, j\neq i}^n \boldsymbol{\mathcal{D}}_j = \emptyset$. Thus $\sigma(\alpha(d)) \in \boldsymbol{\mathcal{D}}_i$ or $\sigma(\alpha(d)) \in \mathcal{SD}$. Written in terms of permutation φ we obtain:

$$\varphi(d) \in \boldsymbol{\mathcal{D}}_i \text{ or } \varphi(d) \in \mathcal{SD}$$

We can deduce from the above formula, that $\varphi^*(d)$ intersect $\mathcal{SD}$ or is included in $\boldsymbol{\mathcal{D}}_i$.

If $\varphi^*(d) \subset \boldsymbol{\mathcal{D}}_i$ we have $\varphi^*(d) = \boldsymbol{\mathcal{D}}_i$ since the tree $\boldsymbol{\mathcal{D}}_i$ has only one φ-orbit. Using Lemma 6 we have:

$$\sigma^*(\varphi^*(d)) \cap \mathcal{SD} = \sigma^*(\boldsymbol{\mathcal{D}}_i) \cap \mathcal{SD} \neq \emptyset$$

We can thus consider d' in $\varphi^*(d)$ such that $\sigma(d') \in \mathcal{SD}$. Then $\alpha(d') \in \boldsymbol{\mathcal{D}}_i = \varphi^*(d)$ and $\varphi(\alpha(d')) = \sigma(d') \in \mathcal{SD}$. Thus $\varphi^*(d) \cap \mathcal{SD} = \varphi^*(\alpha(d')) \cap \mathcal{SD} \neq \emptyset$. This is forbidden by our hypothesis $\varphi^*(d) \subset \boldsymbol{\mathcal{D}}_i$.

□

Definition 40 Connecting series

Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a contraction kernel $F = (\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n)$ and a dart $d \in \mathcal{SD}$, the **connecting serie** *associated to d is equal to:*

$$CS(d) = d, \varphi(d), \ldots, \varphi^{n-1}(d) \text{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\}$$

Remark 3 *The connecting serie is defined for all dart d in $\mathcal{SD}$ since the set $\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\}$ contains at least $|\varphi^*(d)|$.*

Note that we do not talk of connecting paths, since the serie $CS(d)$ is not always a path according to definition 12 (see Figure 13). If $CS(d)$ is a path it connects the vertex $\sigma^*(d)$ to the vertex $\sigma^*(\varphi^n(d))$ where d and $\varphi^n(d)$ belong to $\mathcal{SD}$.

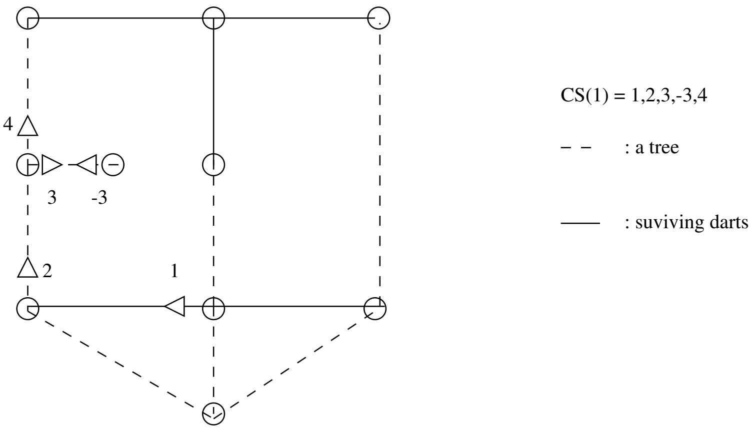


Figure 13: *A connecting serie which is not a path*

Definition 41 Connecting series set

Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a contraction kernel F, the set of connecting serie will be denoted $\boldsymbol{\mathcal{D}}_C$.

Proposition 20 *Given a connected combinatorial map G and a contraction kernel F, the application:*

$$CS \left(\begin{array}{ccc} \mathcal{SD} & \rightarrow & \boldsymbol{\mathcal{D}}_C \\ d & \mapsto & CS(d) \end{array} \right.$$

is bijective.

Proof:

This application is trivially surjective. Now let us consider two darts d and d' in $\mathcal{SD}$ such that $d \neq d'$. Each serie contains one and only one dart in $\mathcal{SD}$. Moreover, $d \in CS(d)$ and $d' \in CS(d')$ involve $d' \notin CS(d)$ and $d \notin CS(d')$. Therefore $CS(d) \neq CS(d')$ □

Proposition 21 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a contraction kernel $F = (\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n)$ we have:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \exists! d' \in \mathcal{SD} \mid d \in CS(d')$$

Proof:

By definition, each connecting serie contains only one dart in $\mathcal{SD}$, thus if d belongs to $\mathcal{SD}$, $CS(d)$ exists and is unique.

Now let us consider $d \in \boldsymbol{\mathcal{D}}_i$ for a given index i. According to proposition 19 we have: $\varphi^*(d) \cap \mathcal{SD} \neq \emptyset$. Let us consider:

$$d' \in \varphi^*(d) \cap \mathcal{SD} \mid d' = \varphi^{-n}(d) \text{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^{-p}(d) \in \mathcal{SD}\}$$

we have obviously $d \in CS(d')$. Let us suppose that we can find an other dart $d'' \in \mathcal{SD}$ such that $d \in CS(d'')$. Then $d'' = \varphi^{-p}(d)$ with $p > n$. Thus the serie $CS(d'') = d'', \varphi(d''), \ldots, \varphi^{p-n}(d'') = d', \ldots, d, \ldots$ contains at least the two darts d and d' in $\mathcal{SD}$, which is forbidden by the definition of a connecting serie. □

7.3 Connecting serie map

Definition 42 Reversal of Connecting series

Given a connected combinatorial map and a contraction kernel F, the opposite permutation α_C from $\boldsymbol{\mathcal{D}}_C$ to itself maps each connecting serie $CS(d)$ with $d \in \mathcal{SD}$ to $CS(\alpha(d))$. More formally,

$$\alpha_C \left(\begin{array}{lll} \boldsymbol{\mathcal{D}}_C & \rightarrow & \boldsymbol{\mathcal{D}}_C \\ CS(d) & \mapsto & CS(\alpha(d)) \end{array} \right.$$

Remark 4 *The function which associates to each dart its connecting serie and the permutation α being bijective α_C is bijective. It is thus a permutation on $\boldsymbol{\mathcal{D}}_C$. Moreover,*

$$\alpha_C \circ \alpha_C(CS(d)) = CS(\alpha \circ \alpha(d)) = CS(d)$$

α_C is an involution.

Lemma 7 *Given a connected combinatorial map and a contraction kernel F, the application:*

$$follow \left(\begin{array}{lll} \mathcal{SD} & \rightarrow & \mathcal{SD} \\ d & \mapsto & \varphi^n(d) \text{ with } CS(d) = d, \ldots, \varphi^{n-1}(d) \end{array} \right.$$

is bijective.

Proof:

The connecting serie $CS(d)$, and thus $\varphi^n(d)$, is defined for all dart in $\mathcal{SD}$. Now let us suppose that we can find two darts d and d' such that $follow(d) = follow(d')$. Then it exists two integers n, p, with $n \geq p$ such that $\varphi^n(d) = \varphi^p(d')$. Thus we have $d' = \varphi^{n-p}(d) \in \mathcal{SD}$. The integer n being the minimal integer different from zero which realizes this equality we have $n = p$ and thus $d = d'$. $\square$

Proposition 22 *Given a connected combinatorial map and a contraction kernel F, the application:*

$$\varphi_C \left(\begin{array}{lll} \boldsymbol{\mathcal{D}}_C & \rightarrow & \boldsymbol{\mathcal{D}}_C \\ CS(d) = d, \varphi(d), \ldots, \varphi^{n-1}(d) & \mapsto & CS(\varphi^n(d)) \end{array} \right.$$

is a permutation.

Proof:

$\varphi_C(CS(d)) = CS(follow(d))$ $\square$

Definition 43 Connecting serie map

Given a connected combinatorial map G and a contraction kernel F, the **connecting serie map** *associated to G and F is denoted GC and is defined by $GC = (\boldsymbol{\mathcal{D}}_C, \sigma_C = \varphi_C \circ \alpha_C, \alpha_C)$*

7.4 Link between connecting series and contraction

Theorem 2 *Given a connected combinatorial map G and a contraction kernel F, the connecting serie map GC is isomorph to the contracted map $G' = G/\cup_{i=1}^{n} \boldsymbol{\mathcal{D}}_i$. More formally:*

$$GC \cong G/\cup_{i=1}^{n} \boldsymbol{\mathcal{D}}_i$$

Proof:

We have:

$$\begin{array}{lll} GC & = & (\boldsymbol{\mathcal{D}}_C, \sigma_C, \alpha_C) \\ G' & = & (\boldsymbol{\mathcal{D}} - \bigcup_{i=1}^{n} \boldsymbol{\mathcal{D}}_i, \sigma', \alpha) = (\mathcal{SD}, \sigma', \alpha) \end{array}$$

Now, let us consider the application $\phi = (\chi, CS)$ from G' to GC such that:

$$\chi : \begin{array}{lll} \sigma' & \mapsto & \sigma_C \\ \alpha & \mapsto & \alpha_C \end{array}$$

Since the application CS is bijective (see proposition 20), ϕ is bijective. Let us show that it is a morphism, thus that:

$$\forall d \in \mathcal{SD} \left\{ \begin{array}{lll} CS(\alpha(d)) & = & \alpha_C(CS(d)) \\ CS(\sigma'(d)) & = & \sigma_C(CS(d)) \end{array} \right.$$

The first equality is given by the definition of the involution α_C. Moreover we have:

$$\begin{array}{lll} \sigma_C(CS(d)) & = & \varphi_C \circ \alpha_C(CS(d)) \\ & = & \varphi_C(CS(\alpha(d))) \\ & = & CS(follow(\alpha(d))) \end{array}$$

The application CS being bijective, the second equality will be demonstrated iff we show that $\sigma'(d) = follow(\alpha(d))$.

We have, by definition $G' = \overline{\overline{G} \setminus \bigcup_{i=1}^{n} \boldsymbol{\mathcal{D}}_i}$. Thus $\sigma' = \varphi' \circ \alpha$ with $\varphi' = \varphi \circ p_{\boldsymbol{\mathcal{D}},\mathcal{SD}}$. Thus $\sigma'(d) = \varphi \circ p_{\boldsymbol{\mathcal{D}},\mathcal{SD}}(\alpha(d)) = \varphi^n(\alpha(d))$ with:

$$n = Min\{p \in \mathbb{N}^* \mid \varphi^p(\alpha(d)) \in \mathcal{SD}\} \tag{2}$$

Moreover, according to Lemma 7, we have $follow(\alpha(d)) = \varphi^n(\alpha(d))$ with n satisfying equation 2. Thus:

$$\sigma'(d) = follow(\alpha(d)) \Rightarrow CS(\sigma'(d)) = \sigma_C(CS(d))$$

□

8 Conclusion

We have defined in this report the theorical framework needed to perform removal or contraction operations on combinatorial maps. The contraction operation is then generalized thanks to the definition of Decimation Parameter and Contraction Kernel. These definitions allow us to design several contractions in parallel.

The definition of a contraction kernel by labeled pyramids is under development. This expected result together with the ones obtain in this report should allow us to study interesting applications of our model such as: segmentation [3, 1, 2, 4], structural matching [13] or integration of moving objects. Finally, the extension of our model to higher dimensional spaces (3D) should be studied.

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