Generalized Conics: properties and applications

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***Abstract*—In this paper the properties of the generalized conics are used to create a unified framework for generating various types of the distance fields. The main concept behind this work is a metric that measures the distance from a point to a line segment according to the definition of the ellipse. The proposed representation provides a possibility to efficiently compute the proximity, arithmetic mean of the distances and a space tessellation with regard to the given set of polygonal objects, line segments and points. In addition, the weights can be introduced for objects, their parts and combinations. This fact leads to a hierarchical representation that can be efficiently obtained using the pixel-wise operations. The practical value of the proposed ideas is demonstrated on an example of applications like skeletonization, smoothing and optimal location finding.**

I. Introduction

Generalized conics are the level set functions that accept infinitely many focal points. Historically, the main interest to this subject comes from mathematics. For example, a multifocal ellipse (or alternatively a polyellipse) is essential in optimization tasks. For every point in space it measures the weighted arithmetic mean distance to the given focal points [1]. The novelty and the educational value of the present paper is to analyse the properties of the generalized conics with the purpose of bringing them to an image processing domain.

The special cases of the generalized conics, an ellipse and a hyperbola, have been studied by Gabdulkhakova et al. [2], [3] and Langer et al. [4]. For instance, the sum of the distances to the foci is constant for all points of an ellipse. This fact enables to create a metric, Confocal Ellipse-based Distance (CED), that computes the distance from a point to a line segment. Incorporating it in the image processing technique called the Distance Transform (DT) [5], makes it possible to create a generalization of the Voronoi Diagram (VD), called Elliptic-Line Voronoi Diagram (ELVD). One of the main observations about the ELVD is an implicit prioritization of the longer edges and acute angles, as opposed to classical VD. With this regard, Langer et al. [4] analysed the effect of various sampling density on the skeletonization of 2D shapes.

In the present paper the research is continued by analysing the effect of explicit weighting of the points, line segments and polygonal objects. By definition, an ellipse and a hyperbola have a pair of focal points. The notion of the generalized conics enables to introduce any number of focal points, that are represented by any type of shape, for example, a point, a line segment or a polygon. This fact makes a basis for a hierarchical representation. When using the DT for representing the distance values distribution, the combination of several clusters is efficiently computed by a pixel-wise sum or minimum operations.

The remaining of the paper is organized as follows. Section II discusses the theory and properties behind the generalized conics: from special cases (ellipse and hyperbola) to more generic (multifocal ellipse and hyperbola). Then, Section III overviews the CED metric as presented in [2]. It enables to compute the Confocal Elliptic Fields (CEF) for the line segments and polygons, and further enrich the definition of the generalized conics (Section IV). Section V discusses the properties of the proposed representations from the perspective of a shape descriptor. The proposed findings have a potential to enrich the existing view on the distances between the objects, especially, in relation to applications and will be discussed in Section VI. Finally, Section VII concludes the paper.

II. Generalized conics

Consider the Euclidean distance, denoted as δ, between the two 2D points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$:

$$\delta(P, Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \tag{1}$$

Definition 1. *An ellipse, denoted as $E(F_1, F_2; a)$, is the locus of points $P \in \mathcal{R}^2$ such that the sum of their distances to the given pair of focal points F_1 and F_2 is constant:*

$$E(F_1, F_2; a) = \{P \in \mathcal{R}^2 | \delta(P, F_1) + \delta(P, F_2) = 2a\} \tag{2}$$

where $a \geq f = \frac{1}{2}\delta(F_1, F_2)$ is the half-length of the major axis of the ellipse, and f is half the distance between the focal points. For $a = f$ the ellipse degenerates into a line segment, whereas for $a = 0$ - into a circle.

Definition 2. *A hyperbola, denoted as $H(F_1, F_2; a)$, is the locus of points $P \in \mathcal{R}^2$ such that the absolute difference between their distances to the given pair of focal points F_1 and F_2 is constant:*

$$H(F_1, F_2; a) = \{\ P \in \mathcal{R}^2 |\ |\delta(P, F_1) - \delta(P, F_2)| = 2a\} \tag{3}$$

Definition 1 and 2 can be generalized to consider the weighted sum/difference of the distances to the focal points to be constant [6]:

$$E_w(F_1, F_2; c) = \{P \in \mathcal{R}^2 | w_1\delta(P, F_1) + w_2\delta(P, F_2) = c\} \tag{4}$$

$$H_w(F_1, F_2; c) = \{\ P \in \mathcal{R}^2 |\ |w_1\delta(P, F_1) - w_2\delta(P, F_2)| = c\} \tag{5}$$

Here the weights w_1 and w_2 are the positive rational numbers, whereas the elliptic curve transforms to an oval. Consider a case illustrated in Figure 1. Given two focal points F_1 and F_2, $w_1 = 1.1$, $w_2 = 0.7$ the isolines represent the distribution of the distance values computed as c.

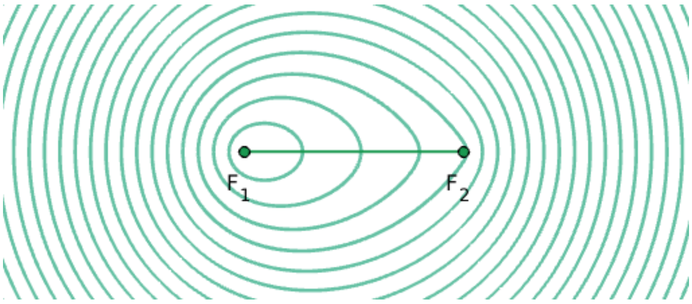


Fig. 1: Isolines of the weighted sums of the distances to the given focal points F_1 and F_2.

In principle, dividing both sides of the Equation 4 by $max(w_1, w_2)$ results in varying a single weighting parameter $0 \leq \alpha = \frac{min(w_1,w_2)}{max(w_1,w_2)} \leq 1$. The corresponding example is shown in Figure 2. After division, the parameter α is multiplied by the distance values of F_2. The transformation of the isoline depending on α is shown using the distinct colors. In the case of $\alpha = 0$, the isoline forms a circle originating from F_1. When $\alpha = 1$, the distance field contains confocal ellipses with the focal points at F_1 and F_2. A special case corresponds to the isoline that passes through the focal point F_2 - it forms a sharp corner. Otherwise, the isoline resembles an oval.

Similar reasoning can be applied to the hyperbola. Consider a division of both sides in Equation 5 by $max(w_1, w_2)$ that results in a parameter $0 \leq \alpha = \frac{min(w_1,w_2)}{max(w_1,w_2)} \leq 1$. Analogically, α is multiplied by the distance values corresponding to the focal point with a smaller weight. Observe the transformation of the isoline depending on the weight of F_2 (see Figure 3). Here, varying α from 0 to 1 changes the curve from circle to the hyperbola branch respectively.

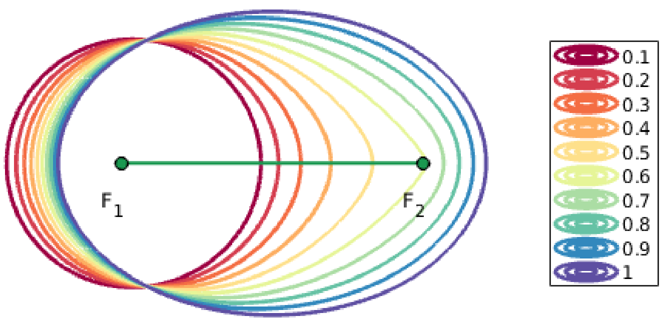


Fig. 2: Evolution of the elliptic isoline depending on α.

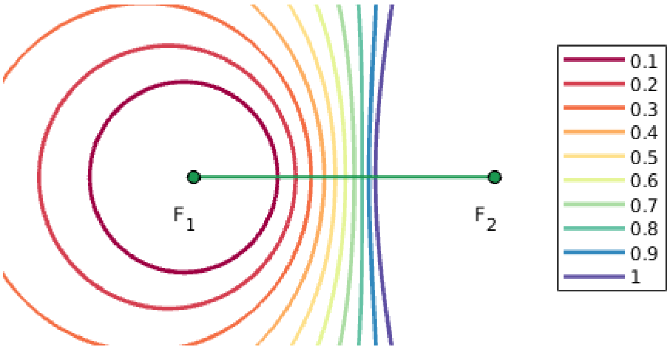
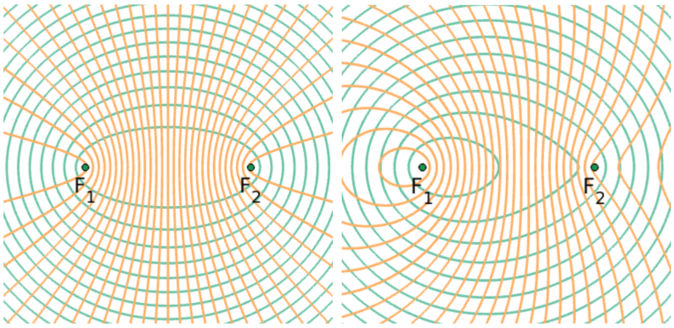


Fig. 3: Evolution of the hyperbolic isoline depending on α.

Definition 3. *A family of ellipses/hyperbolas (with weights or without) that share the common focal points, F_1 and F_2, are called the confocal ellipses/hyperbolas (see Figure 4).*



(a) non-weighted (b) weighted

Fig. 4: A family of confocal ellipses (green) and hyperbolas (orange).

Property 1. *Given any point $P \in \mathcal{R}^2$ there is exactly one curve from the family of confocal ellipses/hyperbolas that passes through it [7]. In other words, a family of confocal ellipses/hyperbolas covers the complete 2D plane:*

$$\bigcup_{c=f}^{\infty} E(c; \alpha) = \mathcal{R}^2 \quad (6) \qquad \bigcup_{c=f}^{\infty} H(c; \alpha) = \mathcal{R}^2 \quad (7)$$

The conic sections can be further generalized into a class of higher-order curves by considering more than two focal points.

Definition 4. *A multifocal ellipse (alternatively, n-ellipse or polyellipse) is a generalization of the ellipse that has N focal points $F_1, F_2, ..., F_N$. It represents a locus of points $P \in \mathcal{R}^2$ with a constant sum of the distances to the N focal points:*

$$ME = \sum_{i=1}^{N} \delta(P, F_i) = const \tag{8}$$

When dividing both sides of the Equation 8 by N, the multifocal ellipses represent the level set functions that average the distance to the given set of focal points [8]. This property plays a crucial role in optimization problems [1].

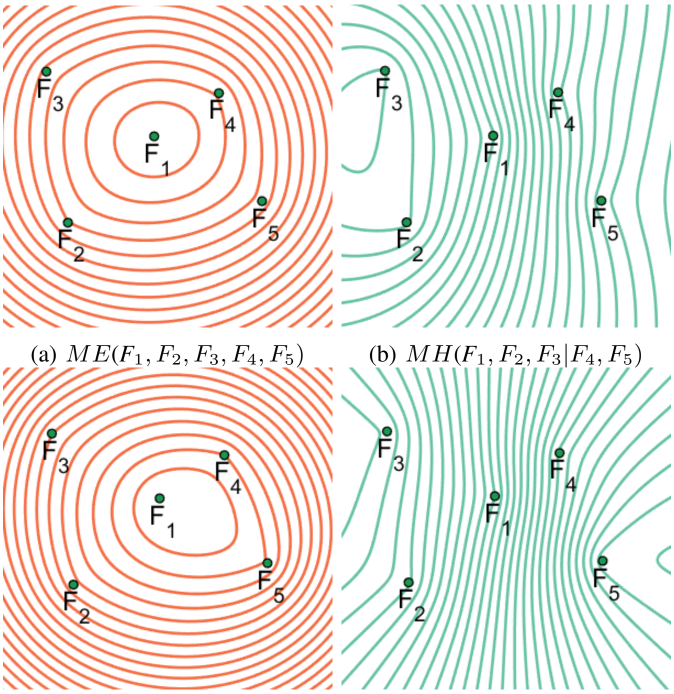
Property 2. *Equation 8 reaches the minimum in exactly one point, when the focal points are non-collinear [9].*

Property 3. *Consider an odd number of ordered collinear focal points. The Equation 8 reaches the minimum at $F_{\frac{N+1}{2}}$ [10].*

Property 4. *Consider an even number of ordered collinear focal points. The Equation 8 reaches the minimum in all the points of the line segment $F_{\frac{N}{2}}F_{\frac{N}{2}+1}$ [10].*

Definition 5. *A multifocal hyperbola is a generalization of the hyperbola defined on two sets of focal points F and G that have M and N elements correspondingly. It represents a locus of points $P \in \mathcal{R}^2$ such that the following absolute difference of the distances remains constant:*

$$MH = |\sum_{i=1}^{M} \delta(P, F_i) - \sum_{j=1}^{N} \delta(P, G_j)| = const \quad (9)$$



(a) $ME(F_1, F_2, F_3, F_4, F_5)$ (b) $MH(F_1, F_2, F_3|F_4, F_5)$

(c) $ME_w(F_1, F_2, F_3, F_4, w_5F_5)$ (d) $MH_w(F_1, F_2, F_3|F_4, w_5F_5)$

Fig. 5: A family of multifocal confocal ellipses and hyperbolas: (a)-(b) non-weighted, (c)-(d) weighted.

Analogically to the case with two focal points, it is possible to introduce the weighting scheme for the multifocal ellipse and hyperbola correspondingly:

$$ME_w = \sum_{i=1}^{N} w_i \delta(P, F_i) = const \quad (10)$$

$$MH_w = |\sum_{i=1}^{M} w_i \delta(P, F_i) - \sum_{j=1}^{N} \nu_j \delta(P, G_j)| = const \quad (11)$$

As can be seen from the Equations 10 and 11, multifocal hyperbola can be considered as a multifocal ellipse whose weights have the different signs.

Property 5. *The interior of the multifocal ellipse is convex [6].*

The families of multifocal confocal ellipses and hyperbolas are illustrated in Figure 5. Given a set of five foci F_1, F_2, F_3, F_4 and F_5, the unweighted sum of their distances to the points in space is shown in Figure 5a. Adding a weight to the point F_5 modifies the distance field by moving the global minimum towards it (see Figure 5c). For multifocal hyperbolas, it is important to define two sets of focal points. In this example, first set contains F_1, F_2, F_3 and the second - F_4, F_5. Figure 5b corresponds to the unweighted focal points, thus, the shift of the isolines is caused by the density of the points. Figure 5d illustrates the multifocal confocal hyperbolas, when the point F_5 has a weight.

III. Confocal Ellipse-based Distance

With regard to the Property 1 of confocal ellipses, it is ensured that each point on the 2D plane has a single distance value (equal to $2a$) associated with it. This can be used to measure the distance between the ellipses [2].

Definition 6. *Consider two confocal ellipses, $E(a_1)$ and $E(a_2)$. Then, the Confocal Ellipse-based distance (CED), $e : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$, can be defined as an absolute difference between the lengths of the semi-major axes a_1 and a_2 of these ellipses:*

$$e(E(a_1), E(a_2)) = |a_1 - a_2| \quad (12)$$

Lemma 1. *CED is a metric.*

A specific case of CED is related to measuring the distance from a point to a line segment. An ellipse has an important geometric property - it can degenerate into a line segment, when the eccentricity value is equal to 1. In Equation 12 consider one of the ellipses, $E(a_2)$, to be a line segment connecting the focal points F_1 and F_2, i.e. $a_2 = f_2$. Then, for each point on an ellipse $E(a_1)$ the distance to the line segment $\overline{F_1F_2}$ with regard to CED equals:

$$e(E(a_1), E(a_2)) = |a_1 - f_2| \quad (13)$$

An alternative representation of the Equation 13 considers the Definition 1 of the ellipse:

$$e(P, l) = \frac{\delta(P, F_1) + \delta(P, F_2) - \delta(F_1, F_2)}{2} \quad (14)$$

Here, P is a point in $\mathbf{R}^2$ and l is a line segment defined by the two end points F_1 and F_2. The distribution of the distance values forms the confocal ellipses with zero values on the line segment $\overline{F_1F_2}$. In contrast to Hausdorff Distance, it takes only the end points of the line segment (taken as focal points) to compute the distance to P. This fact simplifies the computational costs by $N - 2$ operations, where N is the number of points belonging to the line segment. Since using a combination of Euclidean distances, CED can be extended to higher dimensions.

IV. Distance Fields Using the Properties of the Generalized Conics

Previous section discussed a metric, CED, that computes the distance between the points belonging to the confocal ellipses [2]. In order to provide an intuitive explanation of the continuous distance value distribution, we will refer to a classical image processing approach called the Distance Transform (DT) [5].

Definition 7. *Consider a binary image B and a set of the seeds $f \subset B = [0,n] \times [0,m] \subset \mathcal{Z}^2$. Distance Transform (DT) is an operator that converts a binary image into a gray-scale image, $D : B \mapsto \mathcal{R}$. The value of each pixel equals its distance to the nearest seed with regard to the selected metric: $D(M) = \min\{\delta(M,F) | F \in f\}$.*

CED metric in Equation 14 takes the absolute difference between a_1 and f_2. In this regard the confocal ellipses in terms of DT can be defined as follows.

Definition 8. *Consider a line segment defined by the two seeds, $f = \{F_1, F_2\}$, that represent the focal points. Confocal ellipses can be defined as the distance field C_f, where each pixel is mapped to a CED value with respect to the focal points F_1 and F_2:*

$$\begin{aligned} C_f(M) = D_{F_1}(M) + D_{F_2}(M) - D_{F_1}(F_2) = \\ = D_{F_1}(M) + D_{F_2}(M) - D_{F_2}(F_1) \end{aligned} \tag{15}$$

Here the notation $D_X(Y)$ means the value of the pixel Y in the distance field generated from the pixel X. Analogically, let us define the distance field comprised of confocal hyperbolas.

Definition 9. *Let $f_1 = \{F_1, F_2\}$, and $f_2 = \{F_3, F_4\}$ be the seed sets representing two line segments. The distance field $H_{f_1 f_2}$ containing the confocal hyperbolas assigns to each pixel $M \in B$ the difference of its values in the distance fields C_{f_1} and C_{f_2}:*

$$H_{f_1 f_2}(M) = C_{f_1}(M) - C_{f_2}(M) \tag{16}$$

Note, in contrast to Equation 9, the point M of the confocal hyperbola $H_{f_1 f_2}$ can be associated with the negative value. It means, that the point M is closer to f_1 than to f_2.

Inclusion of the weights in relation to Equations 4 and 5, can be implemented by multiplying the corresponding distance fields by w_1 and w_2:

$$\begin{aligned} C_f(M) = w_1 D_{F_1}(M) + w_2 D_{F_2}(M) - w_1 D_{F_1}(F_2) = \\ = w_1 D_{F_1}(M) + w_2 D_{F_2}(M) - w_2 D_{F_2}(F_1) \end{aligned} \tag{17}$$

Let us now define a Confocal Elliptic Field (CEF) [2], that creates a field containing the minimum distances to the given set of focal points.

Definition 10. *Consider a set f representing N line segments and $C = \{C_{f_1}, C_{f_2}, .., C_{f_N}\}$ to be the corresponding distance fields containing the confocal ellipses. The Confocal Elliptic Field (CEF) is defined as a pixel-wise minimum operation that is applied to the distance fields in C:*

$$CEF(M) = \min\{C_{f_i}(M) | i = [1, ..., N], \forall M \in B\} \tag{18}$$

Definition 10 can be extended by adding the weights to the fields of confocal ellipses:

$$CEF(M) = \min\{w_i C_{f_i}(M) | i = [1, ..., N], \forall M \in B\} \tag{19}$$

The essence of the CEF is to enable a creation of the distance fields for complex objects like the collections of points, the line segments, the polygons and to represent their joint minimal proximity to any point in space. Therefore, it is possible to apply the pixel-wise minimum operation to several CEFs (that can be additionally weighted) and generate more complex objects:

$$CEF(M) = \min\{w_i CEF_i(M) | i = [1, ..., N], \forall M \in B\} \tag{20}$$

Figure 6 shows the distance fields comprised of two CEFs of the line segments F_1F_2 and F_3F_4. In Figure 6a the end points and the line segments are not weighted, the isolines depend on the length and mutual arrangements of the line segments. Figure 6b shows the deformation of the distance field due to a weight added to F_1F_2. In Figure 6c, F_1 and F_4 have a larger weight than F_2 and F_3 correspondingly. This field is then further modified by adding an extra non-equal weight to the line segments F_1F_2 and F_3F_4 (see Figure 6d). Generally speaking, the proposed weighting schemes enable to reflect the significance of the objects and/or their parts.

CEF has a specific property, as mentioned in [2]. It implicitly prioritizes the line segments of a greater length by shifting the isolines towards the smaller line segments, and the acute angles over the obtuse angles by pushing the isolines away from the acute angles.

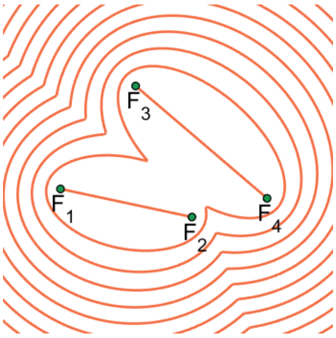
The next group of the distance fields corresponds to multifocal ellipses. With regard to Equation 8, it can be computed as a sum of DTs generated from the focal points $F_1, F_2, ..., F_N$:

$$CMEF = \sum_{i=1}^{N} w_i D_{F_i} \tag{21}$$

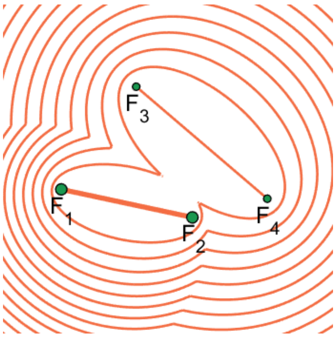
Here CMEF stands for Confocal Multifocal Elliptic Field. For normalization purposes, it is proposed for every pixel to subtract the minimum distance value in CMEF. By Definition 4 the foci are represented by the points. With regard to Definition 10 it is possible to further generalize the notion of CMEF and provide more complex objects as foci, such as line segments or polygons. Though, in this case the convexity of the interior is not guarantied:

$$CMEF = \sum_{i=1}^{N} w_i CEF_i \tag{22}$$

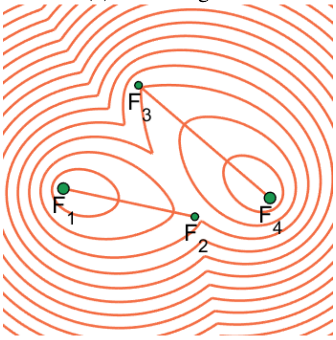
In order to build a hierarchical representation, multiple CMEF distance fields can be combined by the pixel-wise sum operation:



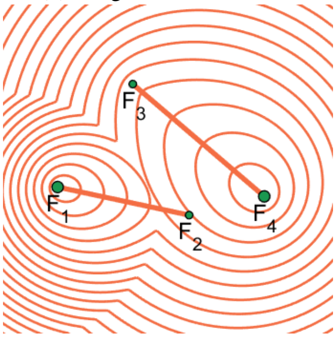
(a) non-weighted



(b) weight added to F_1F_2

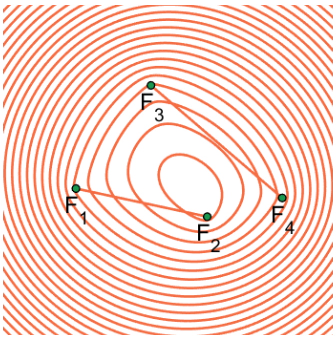


(c) weights added to F_1 and F_4

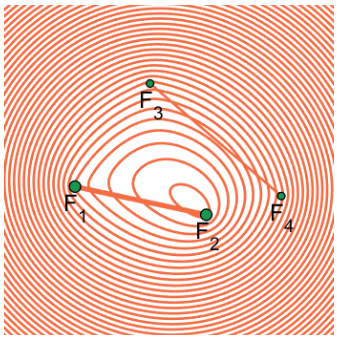


(d) weights added to F_1, F_4, F_1F_2 and F_3F_4

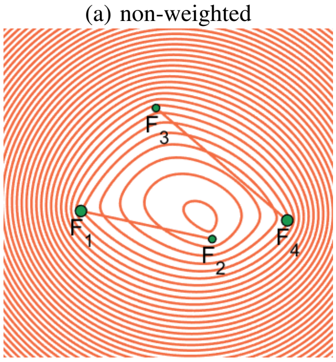
Fig. 6: Examples of CEF of two line segments generated using the various weighting schemes.



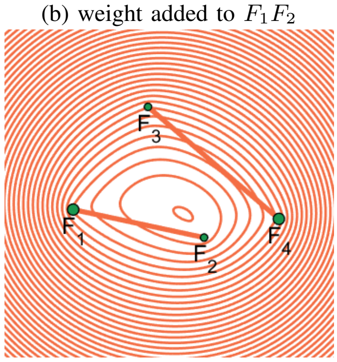
(a) non-weighted



(b) weight added to F_1F_2



(c) weights added to F_1 and F_4



(d) weights added to F_1, F_4, F_1F_2 and F_3F_4

Fig. 7: Examples of CMEF of two line segments generated using the various weighting schemes.

$$CMEF = \sum_{i=1}^{N} w_i CMEF_i \tag{23}$$

Let us now illustrate this statement. Consider a set of line segments F_1F_2 and F_3F_4 from the previous example (see Figure 6). Sum of the corresponding CEFs without any weights according to the Equation 22 is shown in Figure 7a. Similarly to CEF, there is a possibility to weigh an object (for example, a line segment F_1F_2 in Figure 7b), individual points (for example, points F_1 and F_4 in Figure 7c) and both, object and its points (for example, points F_1 and F_4, and line segments F_1F_2 and F_3F_4 in Figure 7d).

The final group of the distance fields described in this paper corresponds to multifocal hyperbolas. As followed from the Definition 5, it needs two sets of objects and can be computed as the pixel-wise difference of their sums. Thus, the Confocal Multifocal Hyperbolic Field (CMHF) can be computed as follows:

$$CMHF = \sum_{i=1}^{M} w_i D_{F_i} - \sum_{j=1}^{N} \nu_j D_{G_j} \tag{24}$$

According to Equation 24, CMHF can also be represented as the difference between the corresponding CMEF fields.

$$CMHF = CMEF_1 - CMEF_2 \tag{25}$$

As can be observed, the zero values in CMHF correspond to the set of points equidistant from CMEF_1 and CMEF_2 in terms of a minimal total average distance. Moreover, the pixel associated with a negative distance value is closer to CMEF_1, while with the positive value - to CMEF_2.

Assume that the focal points are represented by any type of objects, for example, a line segment and a polygon. In this situation, CMHF is the difference between the corresponding pair of CEF fields:

$$CMHF = w_1 CEF_1 - w_2 CEF_2 \tag{26}$$

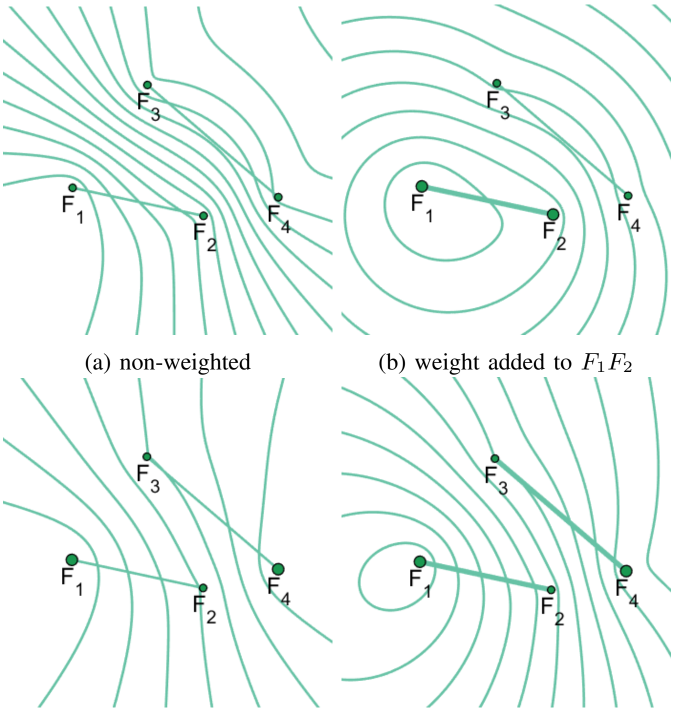
Here, the isolines reflect the proximity to the given pair of objects. In principle, Equation 26 represents rather a hyperbola (see Definition 2), where the focal points are complex objects. When the number of object sets is larger than 2, such hyperbolas are computed for each pair independently.

In general, the essence of CMHF is to find a set of points that have an identical value in several distance fields with regard to some metric.

The CMHF distance fields can be illustrated for the same cases as CMEF. First, consider the given set to be divided into two groups: F_1F_2 and F_3F_4. Then, the CMHF with regard to Equation 25 without any weights is shown in Figure 8a. The weighting schemes are identical to the ones shown in Figure 6 and Figure 7. The corresponding CMHFs fields can be observed in Figure 8b, 8c and 8d accordingly.

V. Discussion

The presented methods for computing the distance fields were visually presented with regard to various types of input



(a) non-weighted

(b) weight added to F_1F_2

(c) weights added to F_1 and F_4

(d) weights added to F_1, F_4, F_1F_2 and F_3F_4

Fig. 8: Examples of CMHF of two line segments generated using the various weighting schemes.

data. In this section the proposed representations are analysed from the perspective of the shape descriptor.

Efficiency. A substantial contribution to the state-of-the-art considers an efficient computation of the distance fields by using a combination of DTs. The existing approaches consider the Hausdorff distance from a point to a line segment, where the latter is represented by the set of its points. The proposed approach takes only a pair of the end points.

Completeness. From the point of representing the space tessellation, the proposed approaches do not have the algorithmic limitations related to the type and number of objects. The formulation enables the extension to higher dimensions.

Invariance. The Euclidean metric that is used in CED can be easily substituted by other distance functions such as Manhattan or Chebyshev distance. Though, in this case the invariance to translation, rotation and scaling will be lost.

Uniqueness. The proposed distance fields uniquely represent the given set of objects. It stems from the fact that the computation is based on the Euclidean distance.

Stability. An important property is related to an ability of preserving the space representation under the changes of the input objects. Especially in a digital domain it relates to inevitable numerical errors that lead to a different position of the sites, their size, or structure [11]. The proposed distance fields are not stable under the motion of the objects. Indeed, the small change in their position implies the small change in the representation. Additionally, the stability is affected by the operations such as insertion and deletion of the seeds.

Abstraction. The proposed method enables the abstraction to multiple hierarchical levels. On one hand, it provides the opportunity to vary the level of sensitivity by changing the weights of the objects. On the other hand, it is possible to create representations that combine existing distance fields.

VI. Applications of Generalized Conics

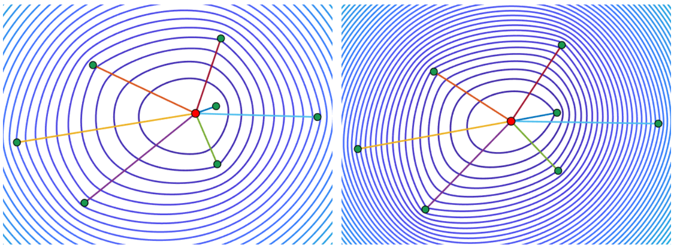
The properties of the generalized conics, that were discussed in this paper, have a practical value. The existing applications include architecture, urban and spatial planning [12], geometric tomography [13].

Gabdulkhakova et al. [2] and Langer et al. [4] use the CEF distance field for skeletonization in 2D. The prioritization of the longer line segments and acute angles was used for smoothing of the noisy shape boundaries [3]. In [3] the hyperbolic distance field was discussed from the perspective of the new Voronoi Diagram type - Elliptic Line Voronoi Diagram (ELVD). Langer et al. [4] proposed a linear in time and memory implementation of ELVD. It is using CED as a metric, and the line segment site is represented only by its endpoints. This reduces the complexity of computing the proximity to all points belonging to the line segment. As opposed to the continuous case the discretization of the space leads to an accuracy problem. For example, there is a possibility of having a two-pixel Voronoi edges that do not have a unique intersection pixel. In the case of Voronoi regions, the accuracy can be defined by $\frac{\sqrt{2}}{2} \times (length\_of\_the\_pixel\_edge)$, which is half of the maximum distance between the centers of two diagonal pixels [4].

A. Optimal Location Problem

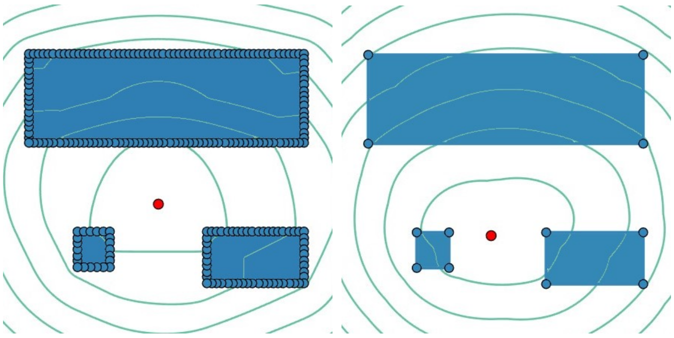
Consider a classical optimization problem: given a set of N point-locations with the positive weights $w_1, w_2, ..., w_N$, find a point that minimizes the sum of the weighted distances to them [14]. Originally it was formulated by P. Fermat who considered only three points of the triangle [1]. In practical domain it is known as a problem of locating the facilities such that the transportation costs are minimized. The existing approaches consider the exact analytical solutions, enumeration of all the possible combinations, approximate statistical and heuristic methods, and linear programming [15], [16]. The complexity of the solution increases with the number of points [17]. This paper uses a discrete approach for computing CMEF with DTs, in order to solve the optimal location problem in linear time.

The formulation of the problem falls exactly into the definition of the multifocal ellipses. The CMEF generated from N seeds reaches the minimum either in one point (see Property 2,4), or in all points belonging to a line segment (see Property 3). An example is illustrated in Figure 9. For $N = 7$ non-collinear focal points the smallest total sum of the distances to them is achieved at the red point (see Figure 9a). The position of such a global minimum can be shifted by introducing the weights to the focal points (see Figure 9b).



(a) non-weighted (b) weighted

Fig. 9: Optimal location solution for the given set of focal points. A red point has the minimum value in the distance fields and is connected to all the focal points.



(a) Hausdorff Distance (b) CED

Fig. 10: Optimal location solution for the given set of polygons. A red point has the minimum value in the distance fields.

According to the Equation 21 and 22, it is also possible to solve the optimal location problem, when the objects are represented by the line segments or polygons (see Figure 10). In the classical Hausdorff Distance case, the resulting distance field is comprised of the sums to all points belonging to the contours of the polygons (see Figure 10a). The global minimum, shown in red, is not influenced by the length of the edges, but the number and density of their points. In contrast, when considering the distance value distribution with regard to CEF (Equation 22), there is an implicit prioritization of the longer edges. Assume the situation illustrated in Figure 10b. As can be observed, the large polygon located on the top of the image pulls the global minimum, shown in red, towards the smaller polygons. Similarly the bottom-right polygon shifts the global minimum towards the bottom-left polygon. Finally, it should be noted, that there are configurations of polygonal objects or line segments, when a single minimum is not guarantied. For example, consider a pair of disjoint line segments F_1F_2 and F_3F_4 that belong to the same line. The minimum is achieved in all points of the line segment F_2F_3.

B. Route planning

The property of edge prioritization becomes useful for another application - route planning. Consider an example in Figure 11. The trajectories are represented by the skeletal points of the medial axis using the classical Hausdorff Distance (red) and CED (green). Let the scene model contain a corridor with several turns and a room. The Hausdorff Distance based route contains the points that are equally distant from the opposite “walls”. In contrast, the CED shifts towards the shorter edges (for example, at the turns) and converges to a classical solution when the edges have a similar length (for example, in a room). As a result the CED-based route represents an optimal solution in terms of the total length of the trajectory.

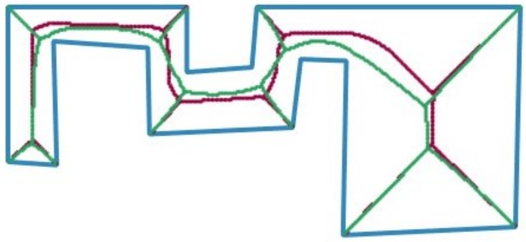


Fig. 11: Comparison of the routes generated using the Hausdorff Distance (red) and CED (green).

VII. Conclusion

The paper introduces a study of the generalized conics and describes them from the image processing perspective. The considered properties are used to create various types of the distance fields. The Confocal Ellipse-based distance (CED) defines the proximity measure between a point and a line segment. The Confocal Elliptic Field (CEF) uses the CED to represent the distance fields of the line segments and the polygonal objects. The Confocal Multifocal Elliptic Field (CMEF) defines for each point in space the arithmetic mean of the distances to the given set of focal points. The Confocal Multifocal Hyperbolic Field (CMHF) defines the closeness of each point in space to one of the focal points. An image processing technique, called the Distance Transform (DT), enables to efficiently compute the CEF, CMEF and CMHF as a combination of pixel-wise minimum, sum and difference operations applied to the distance fields of the given focal points. The properties of the above fields enable to apply various weighting schemes for objects, their parts and groups, and promote a hierarchical representation. This leads to a possibility of using them for solving practical problems like skeletonization, smoothing, optimal location problem and route planning. The idea of measuring the distance from a point to a line segment using the CED, in general, opens a new vision to the classical problems.

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References

[1] Á. Nagy, "A short review on the theory of generalized conics," *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, vol. 31, no. 1, pp. 81–96, 2015.

[2] A. Gabdulkhakova and W. G. Kropatsch, "Confocal ellipse-based distance and confocal elliptical field for polygonal shapes," in *24th International Conference on Pattern Recognition (ICPR)*, 2018, pp. 3025–3030.

[3] A. Gabdulkhakova, M. Langer, B. W. Langer, and W. G. Kropatsch, "Line Voronoi diagrams using elliptical distances," in *Joint IAPR International Workshops on Statistical Techniques in Pattern Recognition (SPR) and Structural and Syntactic Pattern Recognition (SSPR)*, 2018, pp. 258–267.

[4] M. Langer, A. Gabdulkhakova, and W. G. Kropatsch, "Non-centered Voronoi skeletons," in *International Conference on Discrete Geometry for Computer Imagery*, 2019, pp. 355–366.

[5] G. Borgefors, "Distance transformations in digital images," *Computer vision, graphics, and image processing*, vol. 34, no. 3, pp. 344–371, 1986.

[6] C. Groß and T.-K. Strempel, "On generalizations of conics and on a generalization of the Fermat-Torricelli problem," *The American mathematical monthly*, vol. 105, no. 8, pp. 732–743, 1998.

[7] D. Hilbert and S. Cohn-Vossen, "The cylinder, the cone, the conic sections, and their surfaces of revolution," *Geometry and the imagination*, pp. 7–9, 1999.

[8] Z. Melzak and J. Forsyth, "Polyconics. i. polyellipses and optimization," *Quarterly of Applied Mathematics*, vol. 35, no. 2, pp. 239–255, 1977.

[9] P. Vincze and I. Erdös, "On the approximation of convex, closed plane curves by multifocal ellipses," *Journal of Applied Probability*, vol. 19, no. A, pp. 89–96, 1982.

[10] J. Sekino, "n-ellipses and the minimum distance sum problem," *The American mathematical monthly*, vol. 106, no. 3, pp. 193–202, 1999.

[11] D. Reem, "The geometric stability of Voronoi diagrams with respect to small changes of the sites," in *Proceedings of the 27th annual symposium on Computational Geometry*, 2011, pp. 254–263.

[12] M. Petrovic, B. Banjac, and B. Malesevic, "The geometry of trifocal curves with applications in architecture, urban and spatial planning," *SPATIUM International Review*, no. 32, pp. 28–33, 2014.

[13] C. Vincze and Á. Nagy, "On the theory of generalized conics with applications in geometric tomography," *Journal of Approximation Theory*, vol. 164, no. 3, pp. 371–390, 2012.

[14] U. Eckhardt, "Kürzeste Wege und optimale Standorte–von Industriestandorten, Bomben und Seifenblasen," https://www.math.uni-hamburg.de/home/eckhardt/Standort.pdf, 2008, accessed: 2020-10-22.

[15] G. O. Wesolowsky and R. F. Love, "Location of facilities with rectangular distances among point and area destinations," *Naval Research Logistics Quarterly*, vol. 18, no. 1, pp. 83–90, 1971.

[16] G. Rushton, *Optimal location of facilities*. COMPress Wentworth, 1979.

[17] C. Bajaj, "The algebraic degree of geometric optimization problems," *Discrete & Computational Geometry*, vol. 3, no. 2, pp. 177–191, 1988.