Generalized conics with the sharp corners

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Abstract. This paper analyses the properties of the generalized conics which are generated from N focal points with various weights. The weighting enables to obtain up to N corners located at the focal points. The corresponding level sets enable to capture the shape convexities and concavities. From the shape analysis perspective, the generalized conics enrich the variety of shapes that can be described or represented.

Keywords: generalized conics · corner · shape representation

1 Introduction

A generalized conic is a locus of points satisfying the equidistance property of the conic section (parabola, hyperbola or ellipse) that is extended to accept infinitely many focal points. Originally this subject raised an interest in the mathematical community. In particular, a multifocal ellipse (also called n-ellipse or polyellipse) plays a crucial role in solving Fermat-Torricelli [9, 10] and Weber [11] problems. The aim of the present paper is to explore the representational power of the generalized conics for shape analysis.

Our previous research studied the ellipse and hyperbola, their properties and applicability to space tessellation [4], skeletonization [2, 4, 6] and smoothing [6]. The developed framework for efficient computation of the confocal elliptic and hyperbolic distance fields was extended to accept infinitely many weighted foci [3]. Also the geometric nature of the foci was reconsidered to accept not only the points, but also the shapes. This fact enables to apply various weighting schemes for objects, their parts and groups, and promote a hierarchical representation. The application scenarios were further enriched by the facility location problem and route planning. This paper questions the potential for shape representation when using the generalized conics. In particular, the interest lies in analysing the conditions causing the corners in the level sets.

The remaining of the paper is organized as follows. Sec. 2 provides an overview of the main definitions and properties of the generalized conics. Sec. 3 describes

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a method for efficient computation of the corresponding distance fields with the use of Distance Transform (DT) [1, 3]. Sec. 4 and 6 analyse the configurations of the weighted foci that produce convex and concave corners. The discussion continues in Sec. 5 and 7 by introducing an approach to vary the angle at these corners. The possibilities for the shape representation are discussed in Sec. 8. Finally, Sec. 9 concludes the paper.

2 Generalized conics

The properties of conic sections, such as parabola, hyperbola and ellipse, can be extended to accept infinitely many focal points. The resulting level sets are called generalized conics. Each level set depends on the number of focal points, their corresponding weights and the distance metric.

Consider the Euclidean distance, denoted as δ, between the two 2D points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$:

$$\delta(P, Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} \tag{1}$$

Definition 1. *A multifocal ellipse (also referred to as n-ellipse, or polyellipse), denoted as $ME(w_1F_1, w_2F_2, ..., w_NF_N)$, is a locus of points $P \in \mathcal{R}^2$ with a constant sum of the weighted distances to its N focal points:*

$$ME = \sum_{i=1}^{N} w_i \delta(P, F_i) = const \tag{2}$$

Remark 1. The weights corresponding to the focal points in $ME(F_1, F_2, ..., F_N)$ are necessarily positive.

Definition 2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two sets of focal points with M and N elements respectively. A multifocal hyperbola, $MH(w_1F_1, \cdots, w_MF_M | \nu_1G_1, \cdots, \nu_NG_N)$, is a locus of points $P \in \mathcal{R}^2$ such that the following absolute difference of the distances remains constant:*

$$MH = |\sum_{i=1}^{M} w_i \delta(P, F_i) - \sum_{j=1}^{N} \nu_j \delta(P, G_j)| = const \tag{3}$$

Property 1. The level sets ME are convex and compact [7].

Property 2. The equation (2) reaches the minimum in exactly one point, when the focal points are non-collinear [7].

Property 3. If the focal points are collinear, the equation (2) reaches the minimum in all points of a line segment connecting a pair of the focal points [7].

3 Generalized conics from the Distance Transform

To enable an efficient discrete computation of the distance fields representing the generalized conics, it is proposed to use a classical image processing approach called the Distance Transform (DT) [1]. Now coordinates are integers, while Def. 1 and 2 use continuous coordinates. Some of the properties may suffer from sampling, e.g. if the continuous minimum is between two sampling points.

Definition 3. *Let I_{binary} be a 2D binary image and $\mathcal{F} \subset I_{binary}$ - a non-empty set of feature elements: $I_{binary}(F) = 0$ and $I_{binary}(M) = 1$, $\forall F \in \mathcal{F}$, $\forall M \notin \mathcal{F}$. Distance Transform (DT) is an operator that converts a binary into a gray-scale image, $\mathcal{D} : I_{binary} \mapsto \mathcal{R}$. It assigns to each pixel its distance to the nearest feature element with regard to the selected metric d: $\mathcal{D}(M) = \min\{d(M, F) \mid F \in \mathcal{F}\}$, $\mathcal{D}(F) = 0$. DT generated from $\mathcal{F}$ is denoted by $\mathcal{D}_{\mathcal{F}}$.*

According to (2), the multifocal ellipses represent a sum of DTs of the focal points taken as feature elements $\mathcal{F} = \{F_1, F_2, ..., F_N\}$ with their weights:

$$CMEF_{\mathcal{F}} = \sum_{i=1}^{N} w_i \mathcal{D}_{F_i} \tag{4}$$

In (4) $CMEF_{\mathcal{F}}$ denotes the Confocal Multifocal Elliptic Field, defined on an image of the same size as I_{binary}.

The Confocal Multifocal Hyperbolic Field, denoted as $CMHF_{\mathcal{F}|\mathcal{G}}$, can be represented by the difference between a pair of $CMEF$ fields generated from the sets $\mathcal{F}$ and $\mathcal{G}$ (refer to (2) and (3)):

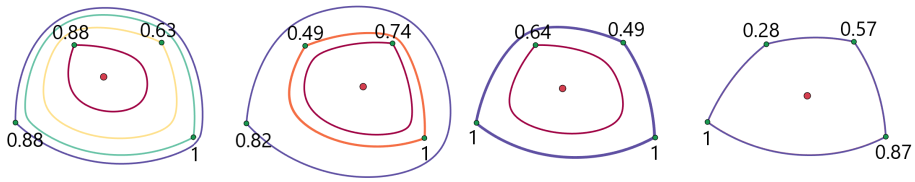
$$CMHF_{\mathcal{F}|\mathcal{G}} = CMEF_{\mathcal{F}} - CMEF_{\mathcal{G}} \tag{5}$$

As opposed to $MH_{w,v}$, pixels in $CMHF_{\mathcal{F}|\mathcal{G}}$ can be mapped to a negative distance value. It means that these pixels are closer to $CMEF_{\mathcal{G}}$ in terms of a minimal total distance. The positive sign defines the pixels that are closer to $CMEF_{\mathcal{F}}$. The zero values - the pixels equidistant from $CMEF_{\mathcal{F}}$ and $CMEF_{\mathcal{G}}$.

4 Multifocal ellipse with corners

This section discusses the configurations and properties of N focal points that generate corners passing through them. By corner we define a point of a curve, where the left-hand tangent differs from the right-hand tangent. In the case of multifocal ellipses the corners are formed by the level sets passing through the focal points. To simplify the upcoming discussion, let us introduce a normalization of N weights $w_1, w_2, \ldots, w_N$, where each of them is divided by the maximum value $max(w_1, w_2, \ldots, w_N)$. As a result, the largest weight becomes 1, whereas the others vary in the open interval between 0 and 1.

Property 4. For the given set of N focal points the weighted multifocal ellipse may contain $1 \leq M \leq N$ corners.



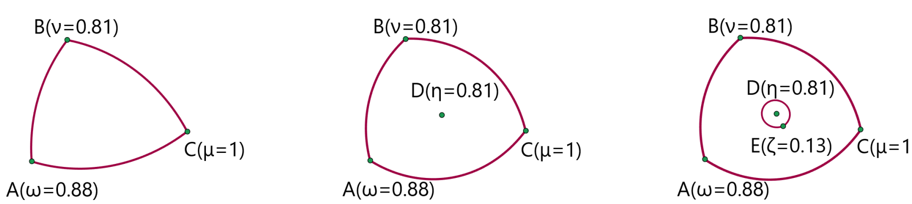
(a) one corner (b) two corners (c) three corners (d) four corners

Fig. 1: Level sets containing one or several corners. The numbers indicate the weights of the focal points.

First, consider N focal points forming a convex hull and their level sets. By taking the different values of weights, a corner is a part of either a level set passing through a single focal point, or is a part of a closed sequence of arcs connecting multiple focal points. An example of a convex hull containing four points is shown in Fig. 1. Here, the global minimum of the distance field is marked as a red point. Changing the weights generates level sets containing one (Fig. 1a), two (Fig. 1b), three (Fig. 1c) and four (Fig. 1d) corners. In the latter, the corner with the larger angle has a focal point with a smaller weight.

Second, let a set of N focal points have at least one inside the convex hull. As stated in Prop. 1, the level sets are convex. So a level set contains maximally as many corners as there are focal points in the convex hull. Let $ME(\omega A, \nu B, \mu C)$ be a multifocal ellipse (Fig. 2a), where $\omega,\ \nu,\ \mu$ are weights. Let us add the points D and E with the weights ν and ζ respectively inside the convex hull (Fig. 2b and 2c). According to the Prop. 1, there can be no level set connecting $A,\ B,\ C,\ D$ and E. Instead, the added points can either be at the global minimum (Fig. 2b), or at the corner of another level set (Fig. 2c).

Property 5. There is a unique combination of the normalized weights that creates a level set passing through all the given focal points.



(a) (b) (c)

Fig. 2: Multifocal ellipses for (a) convex, (b)-(c) non-convex sets of focal points. The numbers indicate the weights.

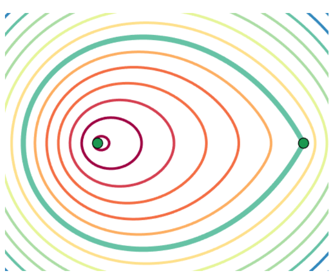
As follows from the axiom about a unique line passing through two points [5], there is a unique polygon that connects N focal points and, thus, a unique set of the respective angles. Similar reasoning can be applied for the angles at the corners. Prop. 5 stems from the fact, that the angle formed at the corner of the level set depends only on the weight of the corresponding focal point. For example, if N focal points form an equiangular polygon, the level set passing through them requires all weights to be 1. In the general case, the larger weight corresponds to a point with a smaller angle.

5 Changing the angle of the egg-shape corner

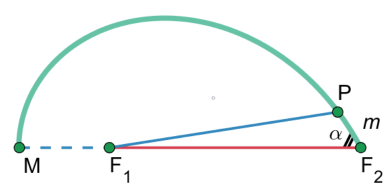
In the previous sections we discussed the multifocal ellipses with regard to the types of level sets containing the corners. One of the findings established a relation between the angles and the weights of the focal points. Here this correspondence is formalized for the level sets conforming an oval, or an egg-shape, generated from a pair of focal points with non-equal weights. According to the Def. 1, the confocal ellipses with two weighted foci F_1 and F_2 can be defined as:

$$ME(w_1F_1, w_2F_2) = \{P \in \mathcal{R}^2 | w_1\delta(P, F_1) + w_2\delta(P, F_2) = const\} \tag{6}$$

Applying the normalization strategy to (6) results in having a single weight $0 \leq \mu = \frac{min(w_1,w_2)}{max(w_1,w_2)} \leq 1$. In special cases, the distance field is composed of concentric circles ($\mu = 0$) and confocal ellipses ($\mu = 1$). Otherwise, the level sets represent an egg-shape with various sharpness. As an example, observe the distance field of multifocal ellipses for $\mu = 0.47$ (Fig. 3a). The particular interest lies in a level set passing through the focal point, thus, having a sharp corner. Let us now define the angle that corresponds to it.



(a) level sets for a pair of weighted focal points



(b) half of the level set containing the corner

Fig. 3: Egg-shape with a sharp corner, $\alpha = 62°$, $\mu = 0.47$.

Theorem 1. *Consider a weighted ellipse, $ME(F_1, \mu F_2)$ with a sharp corner at the focal point F_2. The cosine of the angle α between the major axis and the tangent passing through F_2 equals the weight $\mu < 1$ corresponding to F_2.*

The implication of Th. 1 to shape representation is the enrichment of the geometric primitives that can be used to describe an object or its part. In contrast to ellipses, the generated egg-shapes can fit the corner of the polygonal shape by having one extra parameter for the weight. The angle ϕ of the corner formed by the two tangents at F_2 is $\phi = 2\alpha$. Then the level set satisfies:

$$\delta(F_1, P) + \cos\frac{\phi}{2}\delta(F_2, P) = \delta(F_1, F_2) \tag{7}$$

Given a shape satisfying (7), it is possible to reconstruct its parameters (Fig. 3b). The focal point F_2 is at the corner forming an angle α with the symmetry axis. The point M belongs to the level set of F_2 and is on the same line with F_1 and F_2. Assuming $P = M$ and $\frac{\phi}{2} = \alpha$ in (7) results in:

$$\delta(F_1, F_2) = \frac{\delta(M, F_2) \cdot (1 + cos\alpha)}{2} \tag{8}$$

Consequently, F_1 can be found by moving from F_2 towards M by $\delta(F_1, F_2)$.

6 Multifocal hyperbola with corners

The generalized conics in the form of the multifocal ellipses produce only convex level sets. The concavities can be obtained by using the multifocal hyperbolas. The resultant level sets do not necessarily satisfy the properties of convexity and the location of the global minimum (refer to Prop. 1-3).

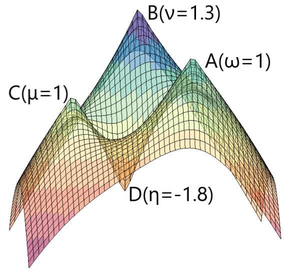
Property 6. There exist no closed level set that passes through all the focal points of the multifocal hyperbola.

The Prop. 6 follows from the fact that a multifocal hyperbola tessellates the space based on proximity to one of the sets of focal points. As a result two sets of focal points are separated by the curve mapped to the zero distance values.

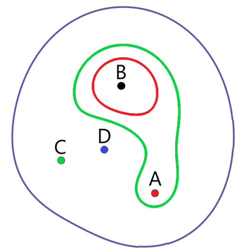
Property 7. The level set that passes through the focal point of the multifocal hyperbola can be represented by the focal point itself and a closed curve.

The weights with the opposite signs cause one group of foci to form the minima, while the other group - the maxima. Thus, the same distance value can be located on the slope of the different focal point(s). Let $MH(\eta D|\, \omega A, \nu B, \mu C)$ be a multifocal hyperbola (Fig. 4). Assume that the weights satisfy the following constraints: $\eta \in [-1 \ldots 0)$; $\mu, \omega, \nu \in (0 \ldots 1]$. The level set containing B is the focal point itself, since it is a global maximum. Point A is a local maximum, hence, its distance value is also present on the slope of B. Similarly, the level set of C contains the focal point itself and a closed curve surrounding A and B. Finally, point D is a local minimum, and the corresponding level set is the point itself and a closed curve surrounding all the focal points.

In general, focal points with the negative weight enable to create concavities in the level sets. As an example, consider a negatively weighted focal point added at the global minimum of the multifocal ellipse with identical weights (Fig. 5). Varying the negative weight value changes the degree of concavity.

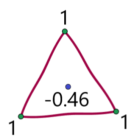


(a) 3D representation of the distance field

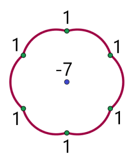


(b) Level sets containing the focal points

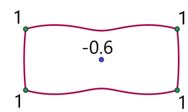
Fig. 4: Level sets of the multifocal hyperbola $MH(\eta D|\ \omega A, \nu B, \mu C)$



(a)



(b)



(c)

Fig. 5: Level sets of the multifocal hyperbola. The numbers indicate the weights.

7 Changing the angle of the hyperbolic corner

Similarly to the egg-shape, it is possible to define the relation between the angle at the corner of the hyperbolic branch and the weight at the corresponding focal point. With respect to Def. 2, the weighted hyperbola generated from two focal points F_1 and F_2 can be formalized as:

$$MH(w_1F_1|w_2F_2) = |w_1\delta(P, F_1) - w_2\delta(P, F_2)| = const \tag{9}$$

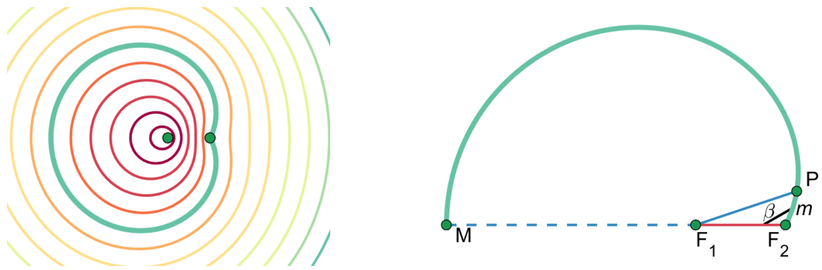
After applying normalization (9), there remains only one weighting parameter $0 \leq \mu = \frac{min(w_1,w_2)}{max(w_1,w_2)} \leq 1$. The level sets of the distance field vary from concentric circles ($\mu = 0$) to hyperbolic branches ($\mu = 1$).

Theorem 2. *Consider a weighted multifocal hyperbola, $MH(F_1|\mu F_2)$, with a sharp corner at the focal point F_2. The cosine of the angle β between the line segment $\overline{F_1F_2}$ and the tangent passing through F_2 equals the weight $\mu < 1$ of F_2 taken with the opposite sign.*

The angle ψ of the concave corner formed by the two tangents at F_2 is $\psi = 2(\pi - \beta)$. Then the level set satisfies

$$\delta(F_1, P) - \cos\frac{\psi}{2}\delta(F_2, P) = \delta(F_1, F_2) \tag{10}$$

The described hyperbolic corners can be potentially used to represent concavities of the shape. By similar reasoning to Sec. 5, consider the hyperbolic shape that



(a) level sets for a pair of weighted focal points

(b) half of the level set containing the corner

Fig. 6: Hyperbolic branch with a sharp corner, $\beta = 118°$, $\mu = 0.47$.

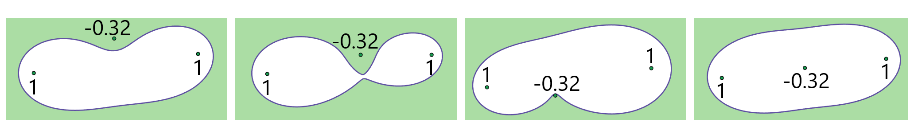
satisfies (10) (Fig. 6b). The focal point F_2 is located at the corner that forms the angle β with the symmetry axis. Substitute $P = M$ and $\frac{\psi}{2} = \beta$ in (10):

$$\delta(F_1, F_2) = \frac{\delta(M, F_2) \cdot (1 - cos\beta)}{2} \tag{11}$$

Consequently, F_1 can be found by moving from F_2 towards M by $\delta(F_1, F_2)$.

8 Shape representation with generalized conics

This paper introduced an approach to represent and reconstruct a shape generated from a pair of focal points with weights (Sec. 5 and 7). It enables to efficiently capture egg-shapes and hyperbolic shapes with corners using three parameters: two end points of the symmetry axis, and an angle at the corner. Such representation is invariant to translation, rotation and scaling. An extension to N focal points is not covered in this paper. Since the number of equations increases with the number of focal points, one possibility to represent a complex shape might be connected with machine learning. The structure of the shape will then be captured in a vector of weights of the focal points.



(a) (b) (c) (d)

Fig. 7: Shapes generated from three focal points. The numbers reflect the weights.

9 Conclusion

This paper introduced the analysis of the generalized conics from the perspective of shape representation. Changing the weights of the focal points in multifocal ellipses enables to generate shapes with convex corners. In turn, multifocal hyperbolas make it possible to capture concave corners. The proposed findings broaden the view on shape representation and have a potential to efficiently generate a complex contour with a small number of focal points (Fig. 7).

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Appendix A

Proof of the Theorem 1

The weighted multifocal ellipse is generated from a pair of focal points F_1 and F_2 such that the weight, $0 < \mu < 1$, corresponds to F_2 (see Fig. 3b). Let us denote $\delta(F_1, F_2) = 2f$, $\delta(F_1, P) = n$, $\delta(F_2, P) = m$, and $\widehat{F_1 F_2 P} = \alpha$. By definition the level set contains a group of points that are mapped to the same distance value. So the distance value at F_2 is identical to the one at P. According to the normalized version of (6), it equals:

$$\delta(F_1, F_2) + \mu\delta(F_2, F_2) = \delta(F_1, F_2) = 2f \tag{12}$$

As can be noted from (12), the distance value corresponding to the level set with a corner equals the length of the line segment F_1F_2. Let us substitute it in the analogical equation for P:

$$\delta(F_1, P) + \mu\delta(F_2, P) = 2f \tag{13}$$

$$n + \mu m = 2f \tag{14}$$

$$\implies n = 2f - \mu m \tag{15}$$

In order to derive an alternative estimate of m, consider a triangle $\triangle F_1PF_2$. According to the law of cosines [8]:

$$m^2 + 4f^2 - 4mf\cos\alpha = n^2 \tag{16}$$

Substituting the value of n from (15) in (16) leads to:

$$m^2 + 4f^2 - 4mf\cos\alpha = 4f^2 - 4\mu mf + m^2\mu^2 \tag{17}$$

$$\implies m = \frac{4f(\mu - \cos\alpha)}{\mu^2 - 1} \tag{18}$$

The important assumption about the point P in continuous space states that it is infinitely close to F_2. This implies that the length of m converges to zero. Then (18) can be further simplified:

$$m = \frac{4f(\mu - \cos\alpha)}{\mu^2 - 1} = 0 \tag{19}$$

$$\implies \mu = \cos\alpha \tag{20}$$

According to (20) the angle formed at the corner of the level set depends on the weight of the focal point and not on the distance between the foci. □

Appendix B

Proof of the Theorem 2

The weighted multifocal hyperbola is defined using a pair of focal points F_1 and F_2 (see Fig. 6b). The level set passing through F_2 contains a sharp corner. Let us denote $\delta(F_1, F_2) = 2f$, $\delta(F_1, P) = n$, $\delta(F_2, P) = m$, and $\widehat{F_1F_2P} = \beta$. Assuming the normalized version of the (9) the distance value at F_2 equals:

$$|\delta(F_1, F_2) - \mu\delta(F_2, F_2)| = \delta(F_1, F_2) = 2f \tag{21}$$

Similarly to the proof for the egg-shape, consider a triangle $\triangle F_1PF_2$ and derive the following relations:

$$n = 2f + \mu m \tag{22}$$

$$m = \frac{-4f(\mu + \cos\beta)}{\mu^2 - 1} \tag{23}$$

In the continuous space m is infinitely small. In the discrete space it can be assigned to zero, resulting in: $\mu = -\cos\beta$. So the angle at the corner of the hyperbolic branch depends only on the weight of the respective focal point. □