Controlling Topology Preserving Graph Pyramids

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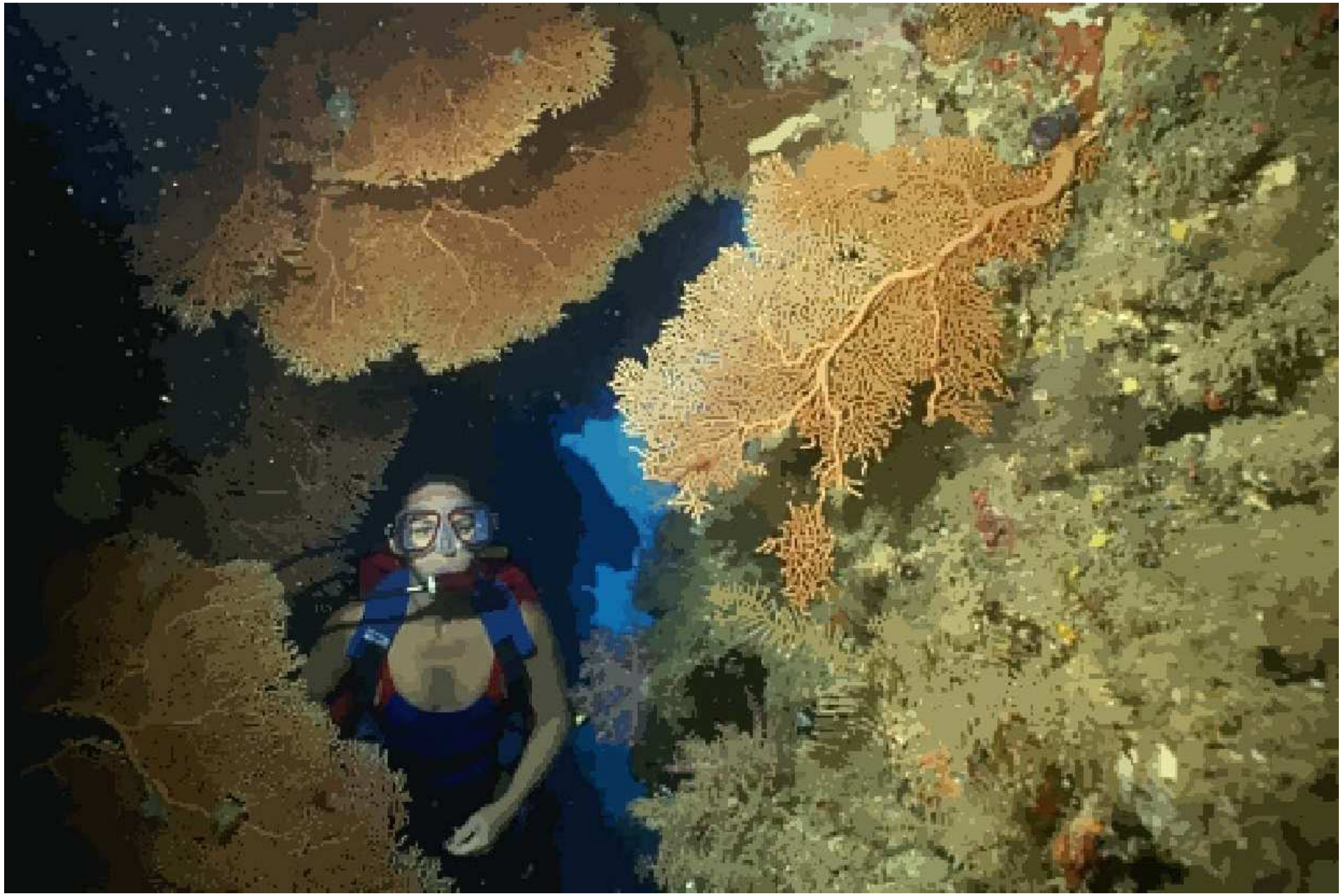
IC PyR+AmId 2022

ICPRAI 2022

Who recognizes the picture Coral?



Who recognizes the picture Cat?

Which one is the original image Fish?

SCIS re-produced with much less colors

Structurally **C**orrect **I**mage **S**egmentation (SCIS)
by preserving the image structure/topology.
Pictures from the Berkeley Image data base [MFTM01]

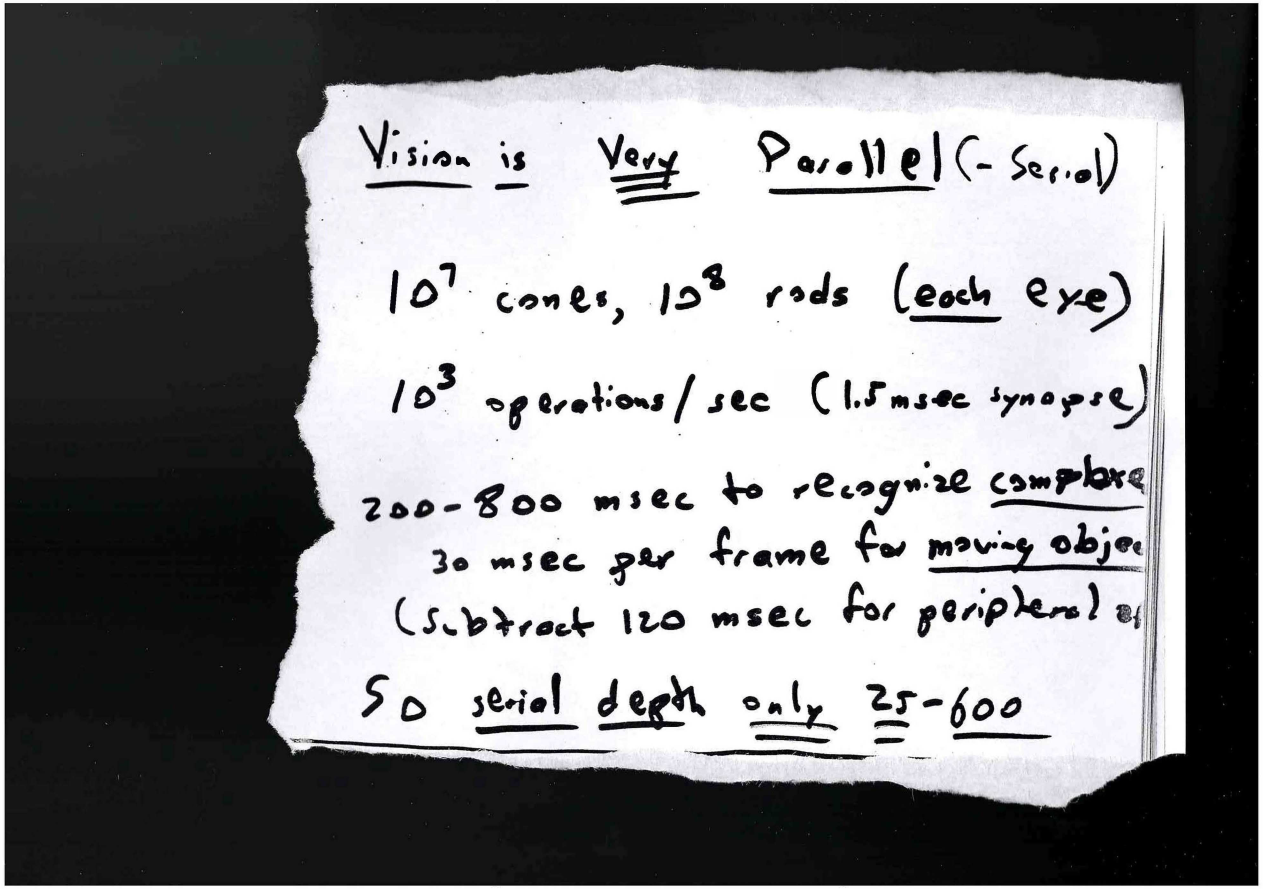
Picture	pixels	regions	reduction by
Coral	154401	12352	92%
Cat	154401	9264	94%
Fish	154401	9264	94%

More in PRIP TR-133, Master Thesis of **Martin Cerman,**
https://www.prip.tuwien.ac.at/publications/technical_reports.php

Overview

- 4 Motivations + Background
- Building LBP-Pyramid + reconstruction
- an INSIGHT problem
- Critical Points from LBP
- Monotonic Path/Curve
- Process + Control + Tasks
- The space between critical points: slopes
- Conclusion and Outlook

1. Biological Perception Problem [Uhr86]



Vision is Very Parallel (- Serial)

10^7 cones, 10^8 rods (each eye)

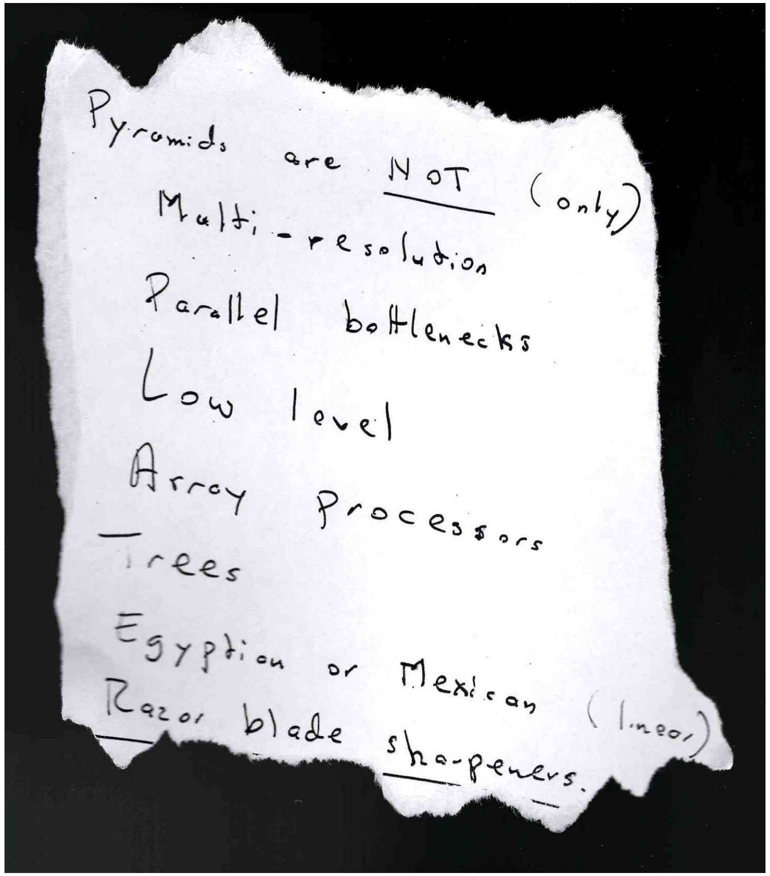
10^3 operations / sec (1.5 msec synapse)

200-800 msec to recognize comple
30 msec per frame for moving objec
(subtract 120 msec for peripheral ef

So serial depth only 25-600

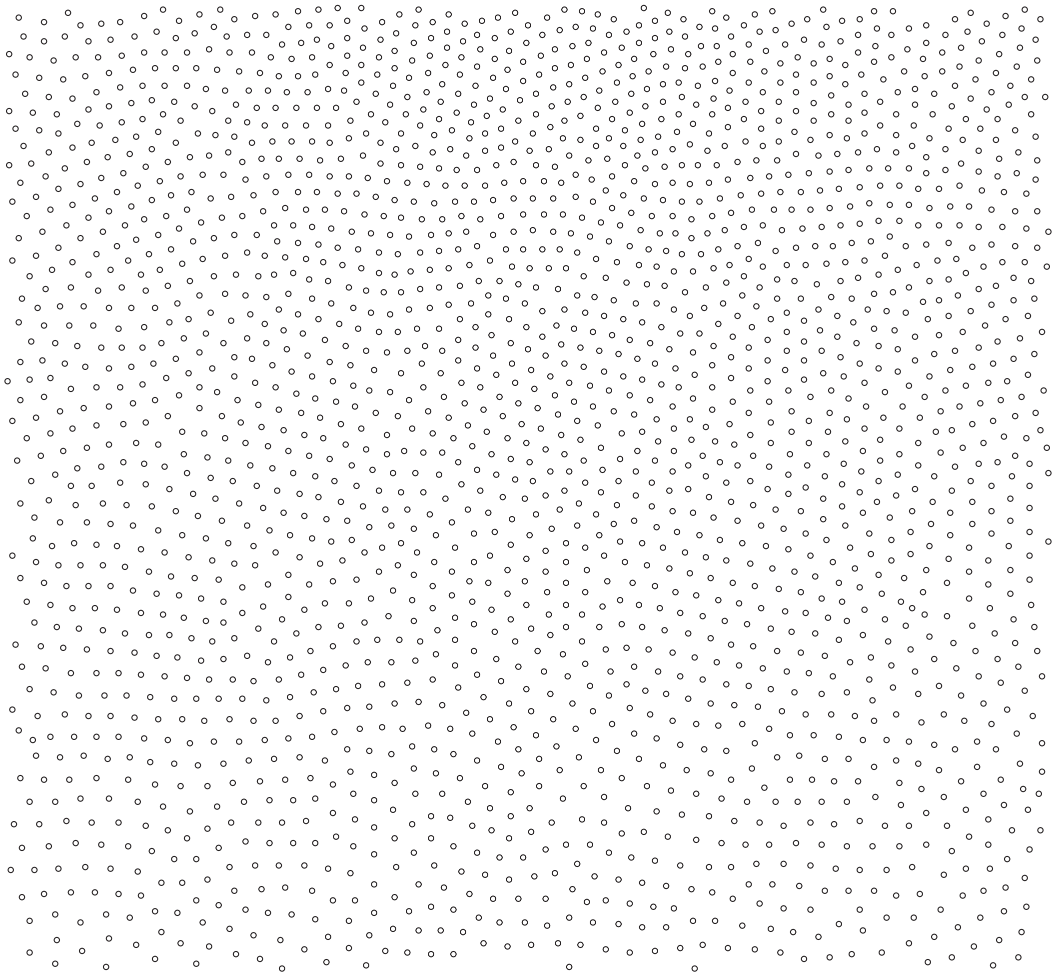
complex vision
in
< 600 serial steps
$\Downarrow$
parallel (MPP)
Log.complexity
$\Downarrow$

Leonard Uhr proposed Pyramids 1986



- "pyramid needs augmentation"
- "... any connected (data-flow) graph could be used."
- "combine bottom-up and top-down"

2. Retina is irregular



. . . NOT ARRAY!

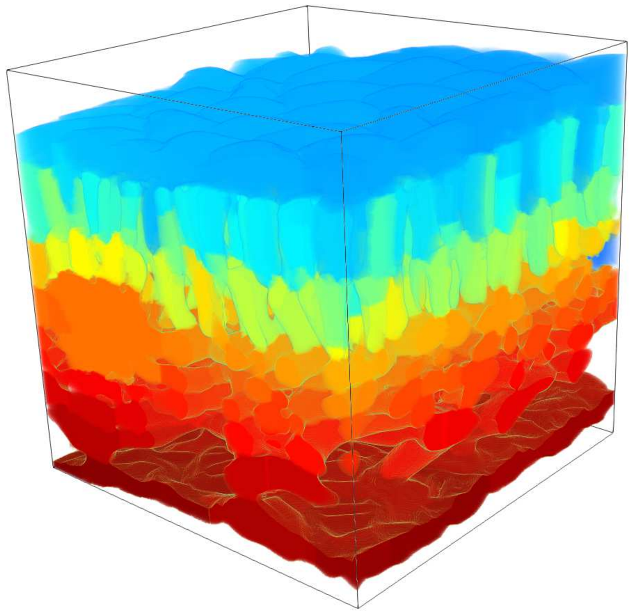
$\Longrightarrow$ graphs

$\Longrightarrow$ irregular embeddings

3. Water's Gateway to Heaven (2020-2024)

https://waters-gateway.boku.ac.at/

- 3D imaging and modeling of transient stomatal responses in plant leaves
- μCT images $2000 \times 2000 \times 2000$, at 2-4 times
- visible objects: different cells, water ways, airspace
- leaves are deformable...
- **understand** opening and closing of stomata for **photosynthesis**



4. Critical/stationary Points

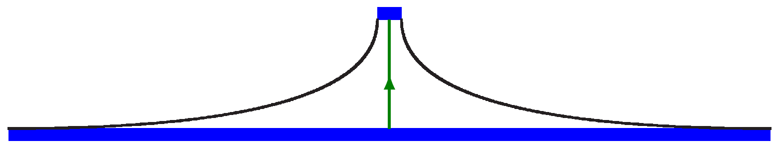
Jan Koenderink [Koe84]: "The Structure of Images".

- Intensity $\Phi(x, y, t)$, (x, y)-coordinates, t scale
- ... generated by convolution with Gaussian kernel $\Phi(x, y, 0) * G(t)$
- **Diffusion** $\Delta\Phi = \Phi_t$ is the basis for scale space theory.
- "Any feature at a coarse resolution is required to possess a 'cause' at finer resolution."
- stationary (critical) points $\Phi_x = \Phi_y = 0$
  Hessian $\Phi_{xx}\Phi_{yy} - \Phi_{xy}^2 \geq 0$... extremum
  Hessian $\Phi_{xx}\Phi_{yy} - \Phi_{xy}^2 < 0$... saddle point

  Alternative: LBP
  no bit switches
  > 2 bit switches
- Extrema and saddle points disappear pairwise when t increases.

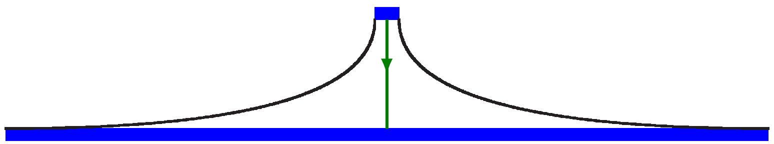
LBP-Pyramid [Cerman 2014] CTIC Timisoara

$|V| = 5000$



Pheasant 154401 Pixels

$|V| = 5000$ parents



$\leq$ 5000 colored regions

bottom-up construction:

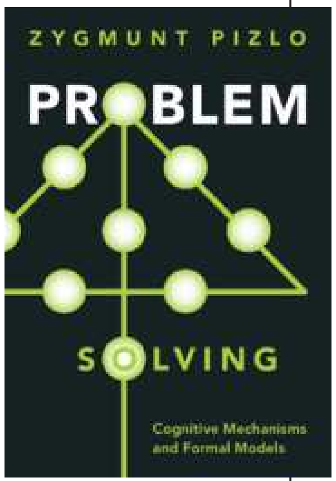
- each level = E-RAG $G = (V, E)$
- edge contraction:
  lowest contrast first
- simplify/remove multi-edges:
  highest contrast first
- preserve relevant critical points,
  determined by LBP

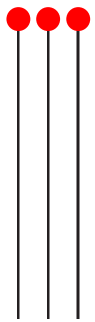
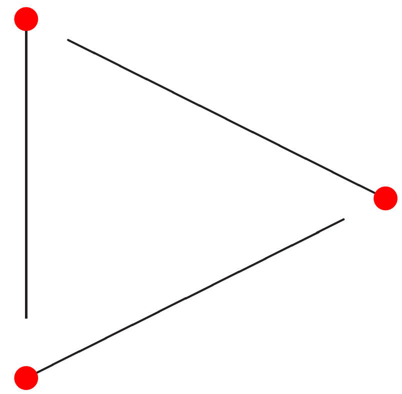
Top-down reconstruction:

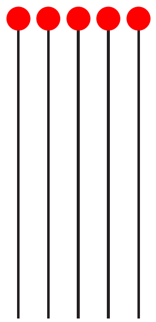
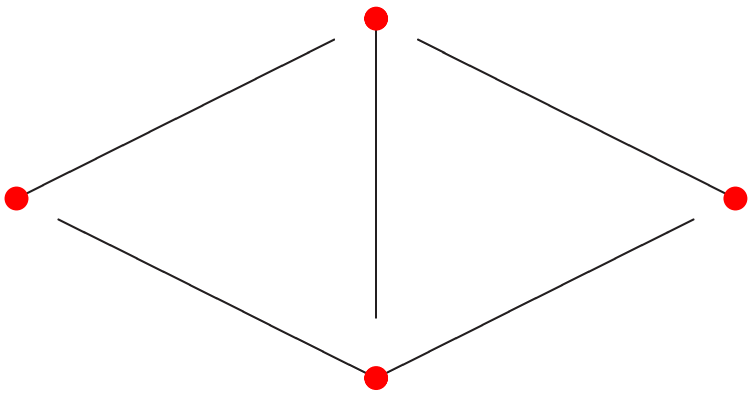
- by canonical representation [TK14]
- edge de-contraction
- (removed) edge re-insertion
- instead of correcting value:
  children inherit parent's value

An Insight Problem from [Piz22]

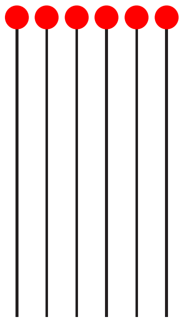
Create n equilateral triangles (△) with m matchsticks:



m=3  $\Longrightarrow$ 1 △: 

m=5  $\Longrightarrow$ 2 △: 

6 Matchsticks give 4 Triangles?

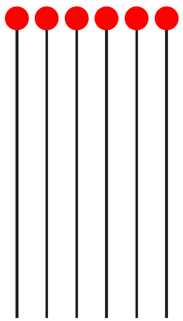
m=6  $\Longrightarrow$ 4 $\triangle$

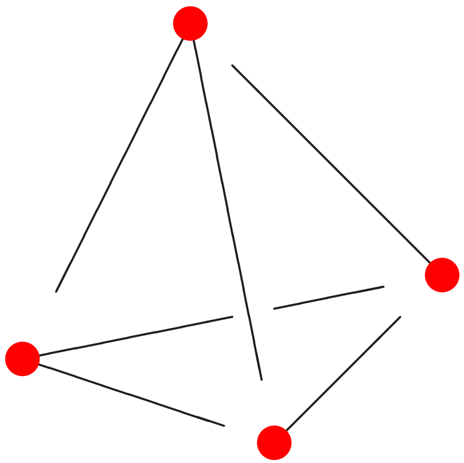
Euler-Poincaré characteristic

#P	-	#E	+	#F	=	1
●	-	m	+	$\triangle$	=	
3	-	3	+	1	=	1
4	-	5	+	2	=	1
?	-	6	+	4	=	1

Insight Problem = difficult

6 Matchsticks give 4 Triangles

m=6  $\Longrightarrow$ 4 $\triangle$:



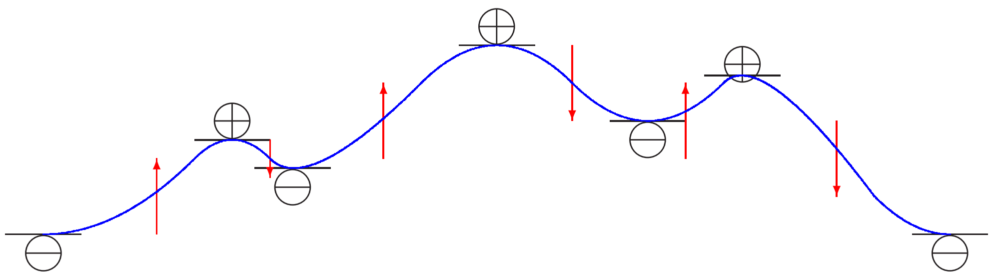
Euler-Poincaré in 3D

#P	-	#E	+	#F	-	#V	=	1
●	-	m	+	$\triangle$	-		=	
4	-	6	+	4	-	1	=	1

1. Solution to insight problem: $\boxed{\textbf{Change representation}}$, 2D $\longrightarrow$ 3D
2. **"AHA"!!!** ...easy to explain.
3. solvable by **optimization? learning?**

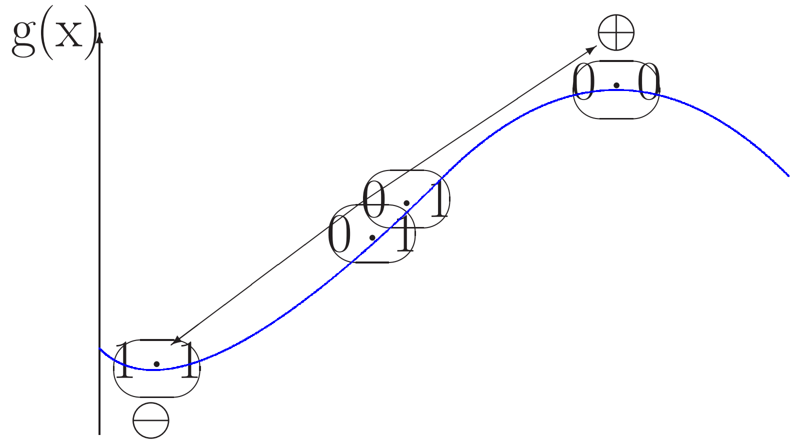
Let's consider LBPs in 1D...

Critical Points of a height Profile



- Critical points in 1D are local extrema
- ... characterized by horizontal tangents
- curves between critical points are **monotonically**
- ... alternating **ups (↑) and downs (↓)**

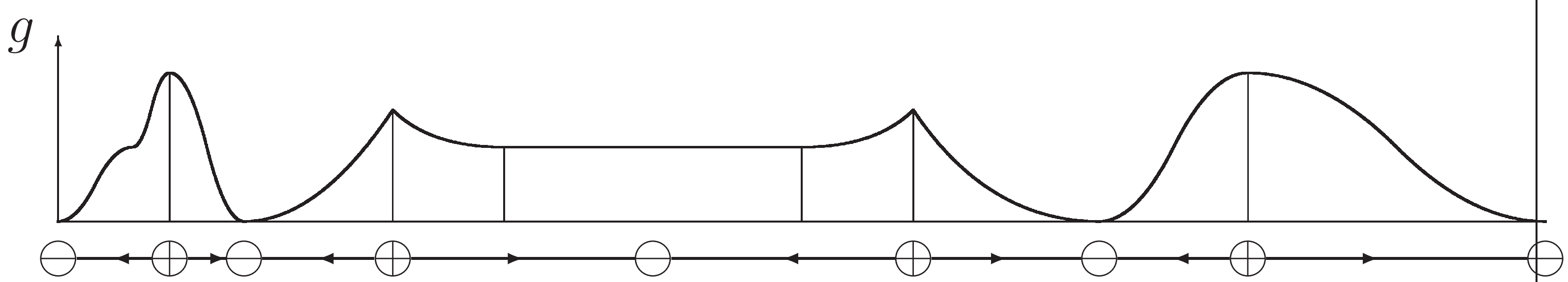
LBPs along monotonic curves



- Local Binary Patterns (LBP) compare a central point with its neighbors,
  0 ... neighbor is smaller
  1 ... neighbor is greater
- along monotonic curves: same ('uniform') LBP-code $0 \cdot 1$
  bounded by local minimum ($\ominus$): $1 \cdot 1$
  and local maximum ($\oplus$): $0 \cdot 0$
- **no derivatives!**

Monotonic Curves/Paths π

1D curve:
$G(V, E)$



- edge/curve between $\oplus \longrightarrow \ominus$ is monotonically decreasing.

  ... **not necessarily smooth!**

- Monotonic path $\pi(p_1, p_n) = (p_1, \ldots, p_n)$:
  $(g(p_{i+1}) - g(p_i))\sigma \leq 0 \quad \forall i \in [1, n-1]$ and given $\sigma \in \{-1, +1\}$.

- **LBP bits $\longleftrightarrow$ oriented edge**

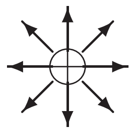
  $\Longrightarrow$ allows LBP for **vertices with different degrees**

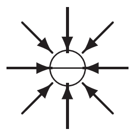
- **CONTRACT edge with lowest contrast**
  $\Longrightarrow$ preserves monotonicity, shortens π, preserves critical points, ...

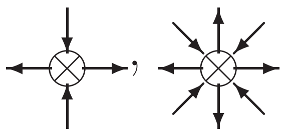
Critical Points in 2D ...

can be recognized also by **(LBP)** Ojala,Pietikäinen,Harwood [OPH96]

Orientation of edges: ② ← ⑧ **downwards**

$\oplus$ local maximum: 

$\ominus$ local minimum 

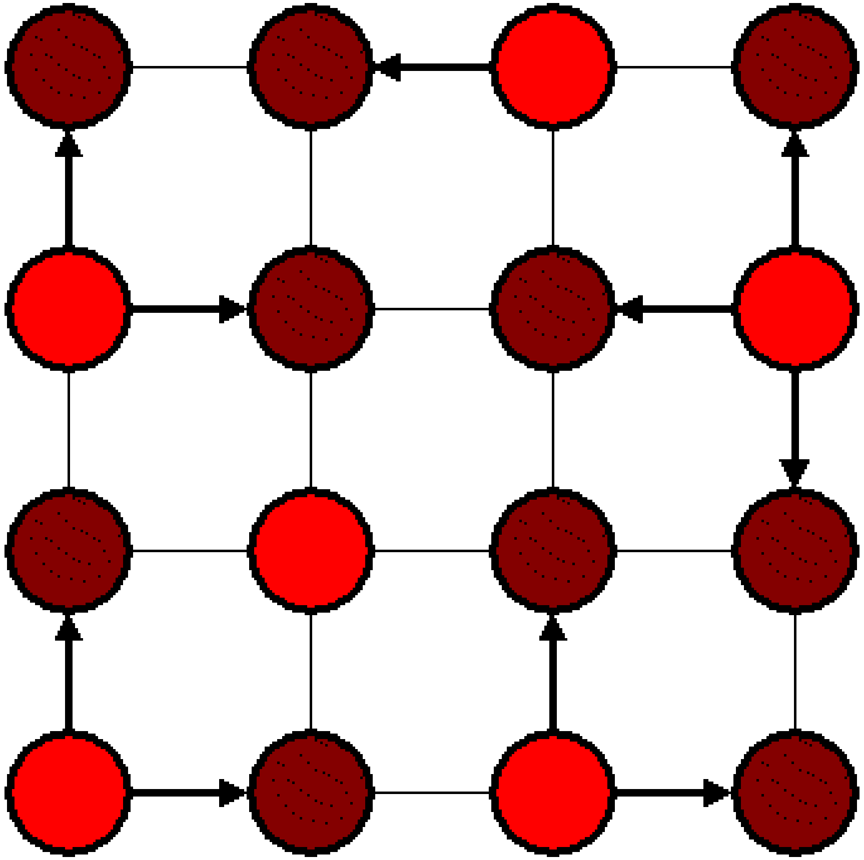
$\otimes$ local saddle  , . . .

To preserve in the hierarchy of the primal graph: $\oplus, \ominus, \otimes$

Lowest contrast pairs $(\oplus, \ominus), (\oplus, \otimes), (\otimes, \ominus)$ can be dropped (as by [Koe84]).

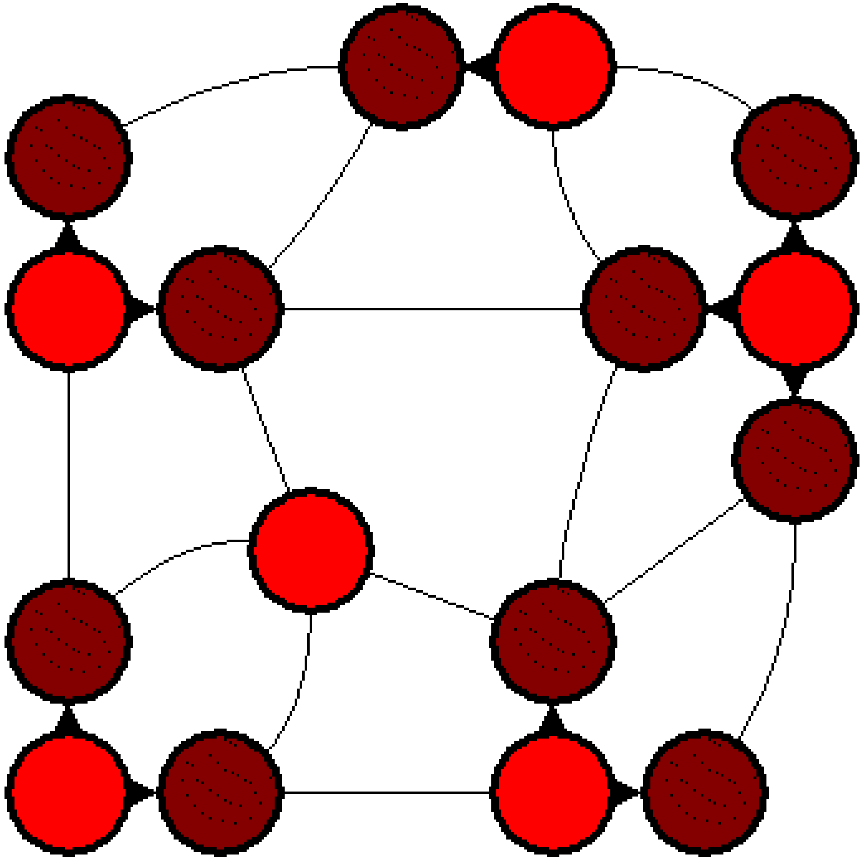
$\Longrightarrow$ **Contraction Kernels**

Contraction Kernels



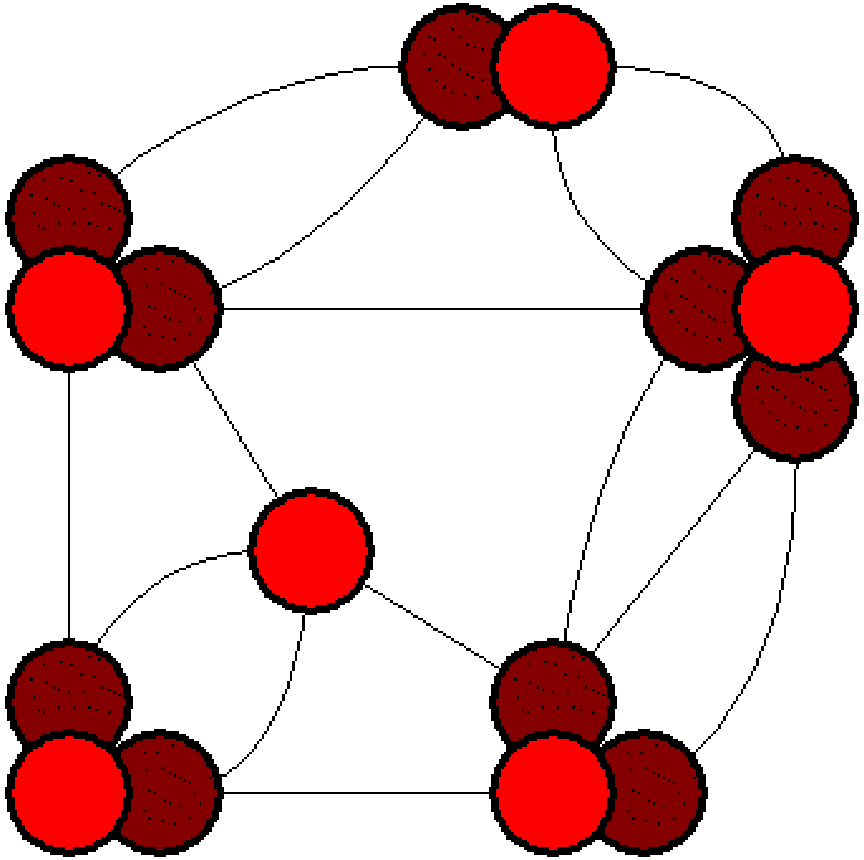
... connect 10 NON-Survivors with 6 Survivors

Edge Contraction



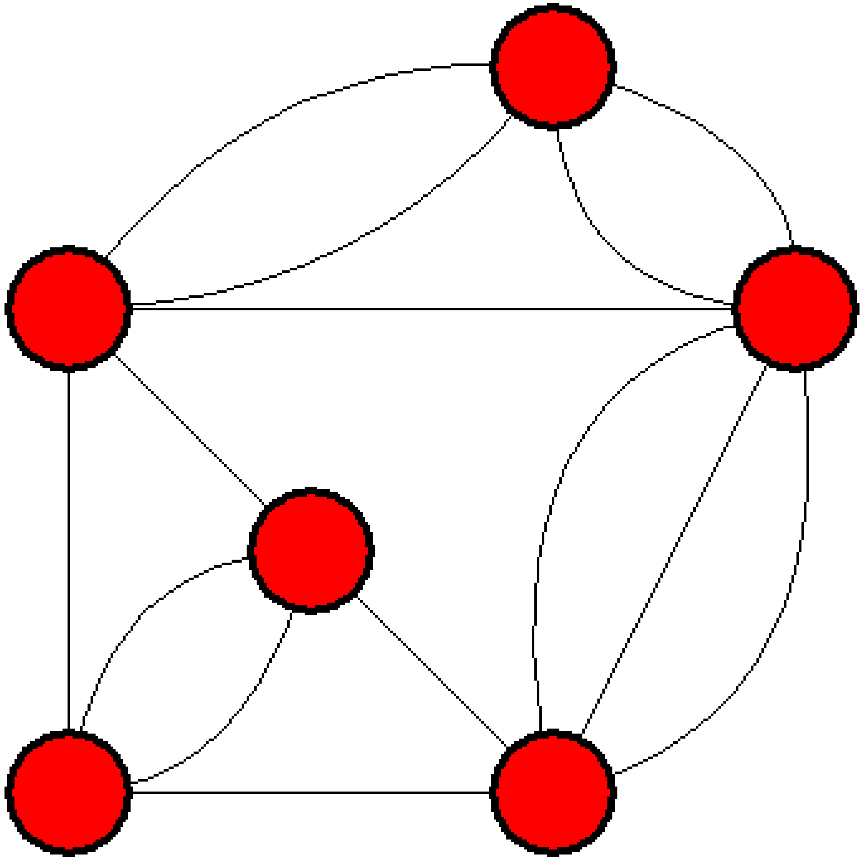
... shrinks 10 selected edges into 6 Survivors

Edge Contraction



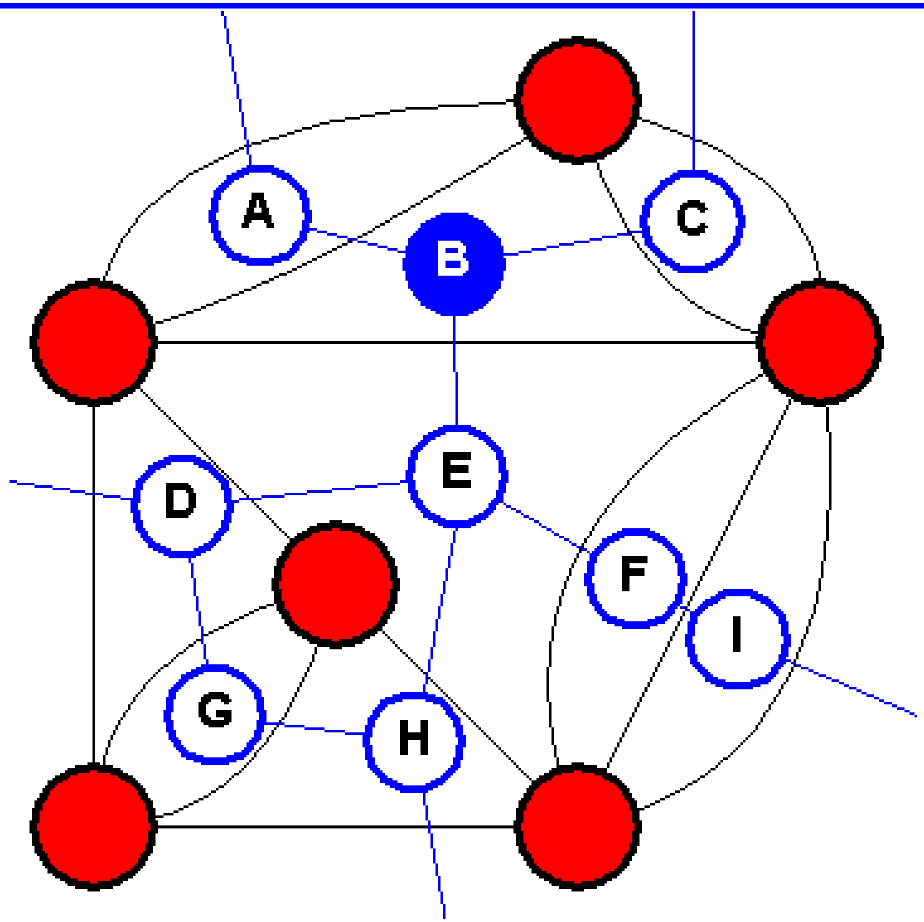
... further shrinks selected edges into Survivors

Contracted Graph



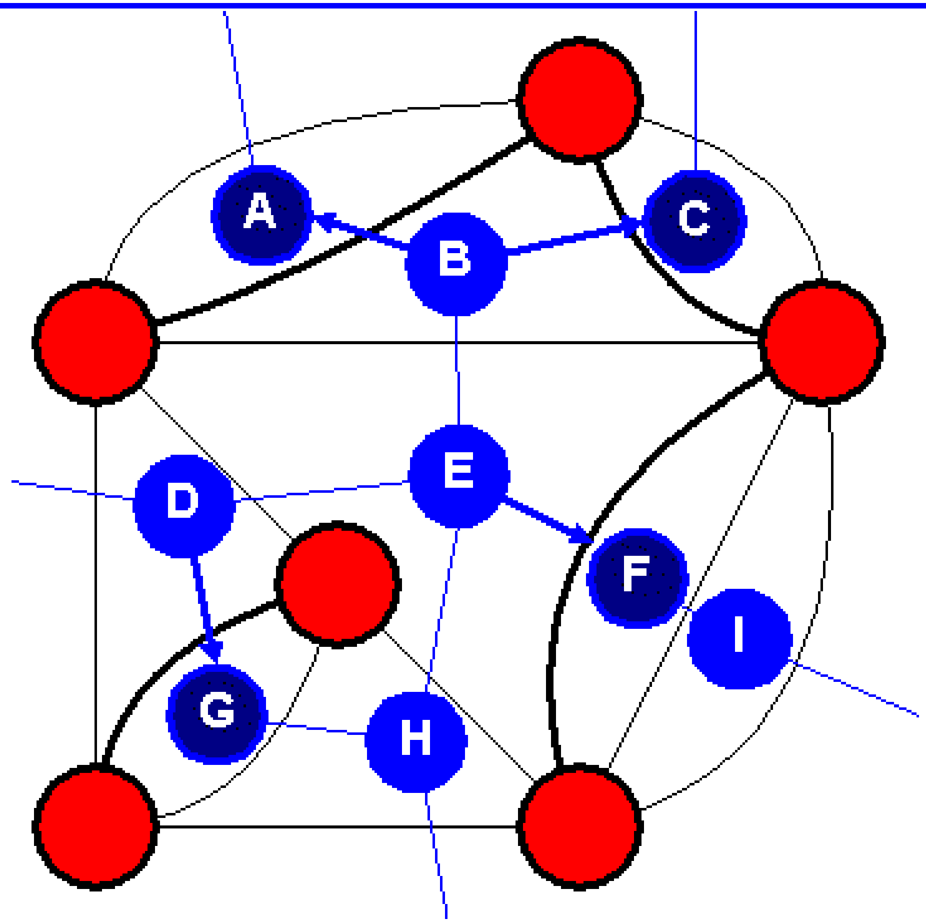
... until they disappear, multiple edges may appear

Simplification: remove multiples?



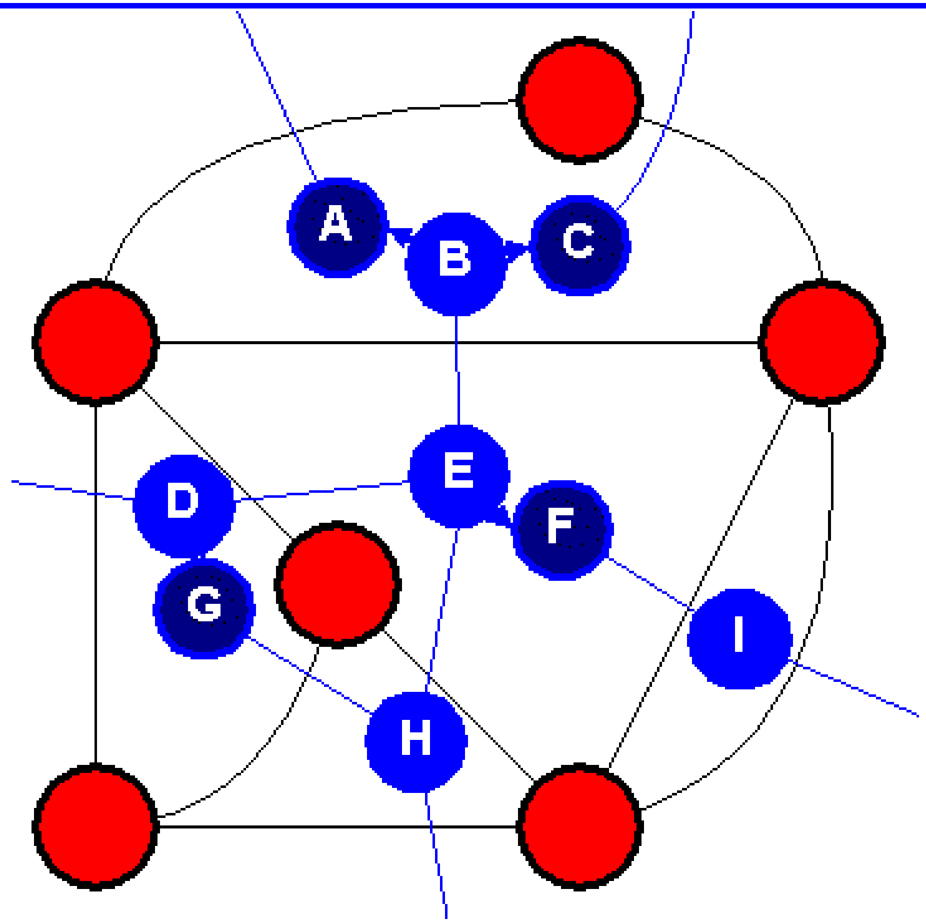
Consider degree of 9 faces (=dual vertices) $A, \ldots, I$

Dual Operations: Contraction vs. Removal



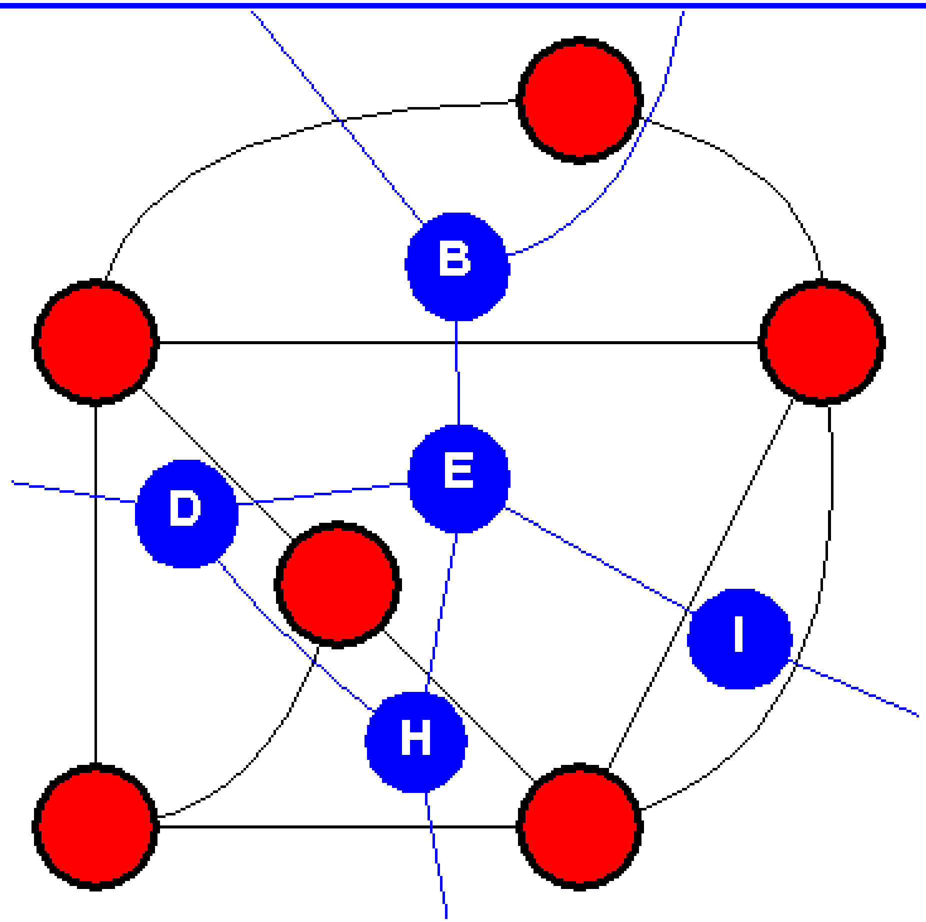
merge degree 2 faces A, C, F, G $=$ **remove 4 double edges**

Simplify: Remove multi-edges = contract duals



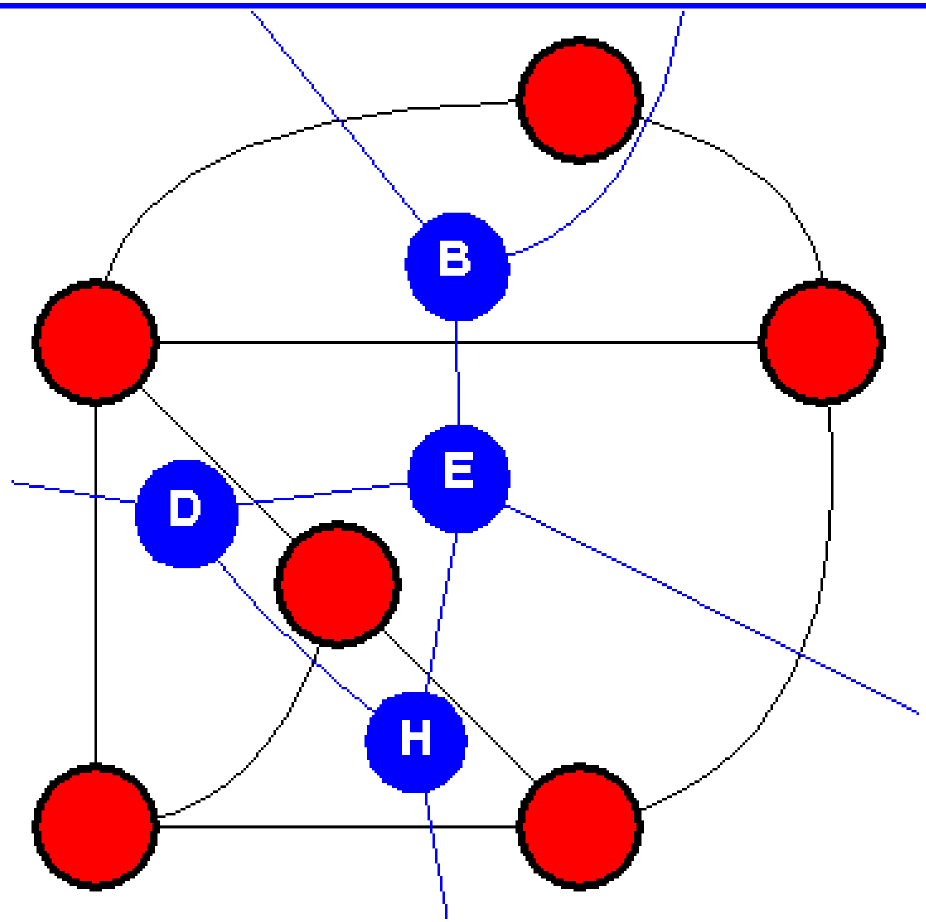
...shrinks 4 edges into dual vertices B, D, E with degree > 2

After 1st Simplify: 1 double edge bounds I



Repeat simplification by merging faces I and E

Dually Contracted Graph



All remaining 4 faces B, D, E, H have degree > 2

BOTTOM-UP CONSTRUCTION

while further abstraction is possible do

1. **select contraction kernels**
2. **perform contraction**
3. **and simplification;**
4. **apply reduction functions → new reduced content**

Each iteration: **new level of the pyramid**

Preserving Topology

Changes Δ by the primitive operations, edge contraction and edge removal:

change of Euler-Poincaré characteristic

operation	$\Delta\#P$	-	$\Delta\#E$	+	$\Delta\#F$	$=0$
contraction	1	-	1	+	0	$=0$
removal	0	-	1	+	1	$=0$

Any number of contractions and removals does NOT change the characteristic!

CONTROL by the CONTENT

1. Selection of contraction kernels
2. Simplification strategies
3. Reduction functions
4. Expansion
   (a) by inverse operations (as in [TK14]):

de-contraction	contraction
re-insertion	removal

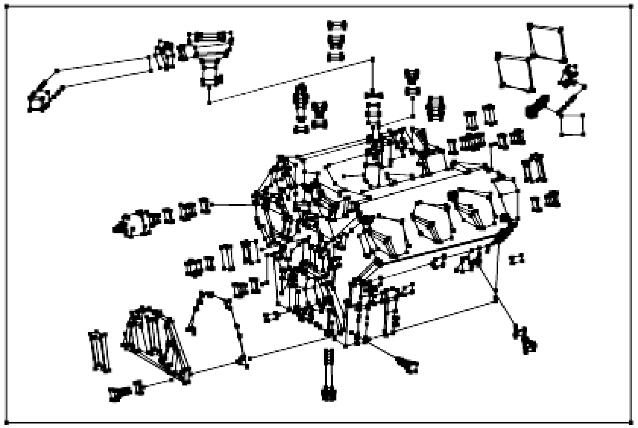
   (b) by interpolation
   (c) by inheritance
   (d) by model refinement

1. SELECT Contraction KERNELs

- RANDOM (like stoch.pyramid → [Mee89])
- Filters → local MAX., the ADAPTIVE Pyramid [JM92]
- RULES (CCL, GAP closing, line drawings [BK99, BK21])
- MATCHING: parametric MODELS (e.g. by correlation)
  'goodness of match' structures (e.g. by GRAPH MATCHING)
- **locally lowest contrast, critical points survive**

2. Simplification Strategies

- complete simplification after each contraction
  ($\mathcal{O}(a^{-1}(n,n))$ $\quad a(n,n)\ldots$Ackermann function)
- only one simplification pass after a contraction
- ALL simplifications after ALL contractions
- content-controlled simplification (some degree 1 or 2 faces survive)
- anticipated simplification before contraction [BBK22]



3. REDUCTION FUNCTIONS

- Inheritance (e.g. for CCL)
- average, convolution filter, **as in DCNN**
- transitive closure (used in line drawings [BK99])
- MODEL name, symbol (like ○ , ● , ⊙ for dotted lines in [Kro95])
  parameters that best match data[HR84].
- LBP: survivors inherit value of critical point
  preserves range of grey values

Selecting Parameters for Abstraction (1)

Application	Important elements survive	Negligible are merged	Redundant are removed
CCL [KM95, MK95, Kro96, BK21]	1 repr/lab	(L, L)	empty faces, deg < 3
segmentation [KB96b, KB96a, KH04a, HMK04]	1 repr/ region	similar, end points	empty faces, deg < 3
2x on curve [Kro97]	X, ends	empty space, connections	empty faces, deg < 3
line images [KBI98, BK98, KB98]	ends, junctions	empty space, connections	empty faces, no touching curve

Selecting Parameters for Abstraction (2)

Application	Important	Negligible	Redundant
matching [PKJ98, GPK02, GPK04]	discrim.template, object boundary	simil.inside object	simil.empty faces, deg < 3
motion [GEK99, MKH04, AIK09]	foreground, static background, articulations	occluded backgr. moving foregr.	empty faces, deg < 3
gap closing [Kro02, KH04b]	1 repr/lab incl. background	(L, L)	empty faces, deg < 3
RAG$^+$ Hierarchy [HK03, HK04, HIK06]	max.ext.Contrast, MST	min.int.Contrast	empty faces, deg < 3
LBP-Pyramid [CGK15, CJGK16, KCBG19a]	critical points, texture, high freq.	lowest contrast	empty faces, deg < 3

Original Berkely Pheasant 154401 Pixels

SCIS(Pheasant) with 30%(154401) Regions

SCIS(Pheasant) with 10%(154401) Regions

SCIS(Pheasant) with 3%(154401) Regions

'Image = Structure + Few Colors', [BGK21]

- most critical points survive
- contracting lowest contrast $\longleftrightarrow$ **preserving high contrast of dual edge**
- In contrast to ALL smoothing reductions:
  preserves high frequencies (small, thin details)
- $\Longrightarrow$ reconstructions with only a few highest levels give good results

…what are the spaces between critical points?

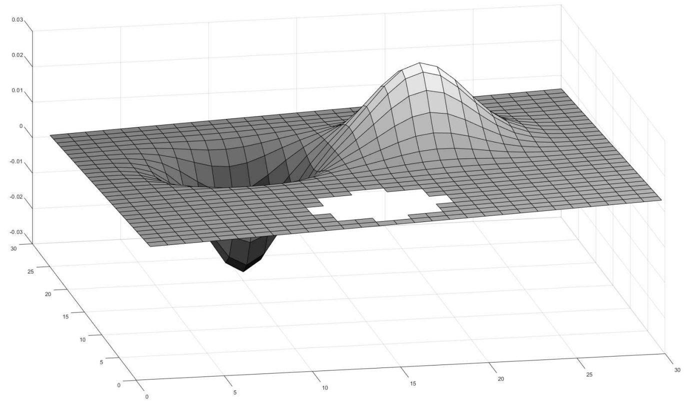
'On the Space Between Critical Points', [KCBG19b]

A connected region R of a 2D surface is a **slope region** iff all pairs of points $\in R$ are connected by a **continuous monotonic curve** $\in R$.

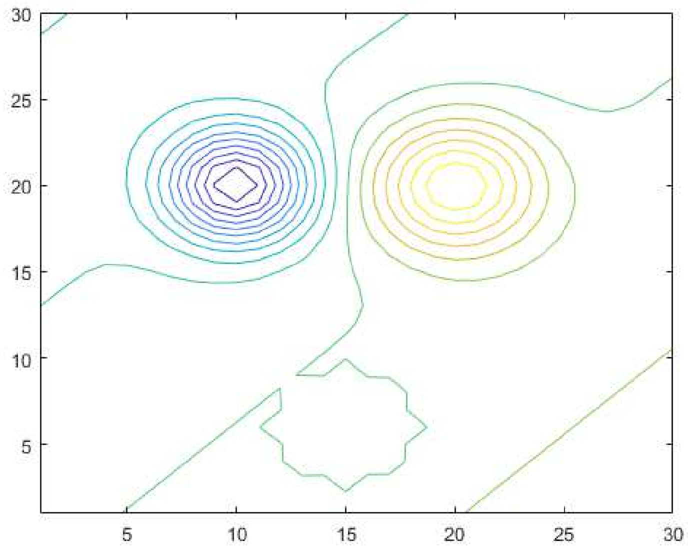
- there may be one $\oplus, \ominus$ inside R
- saddles $\otimes$ have an important role: $\boxed{\otimes \textbf{ are only on the boundary } \partial R}$, never inside R
- more .... [BHK19]

Level Curves

mesh of slope region



bounded by level curve



level curves of slope region.
Extrema $\longrightarrow$ closed level curves.
Hole bounded by level curve.

Slope Complex

An abstract cellular complex is a **slope complex**
if all cells are slope regions

Bounding relations are given by $G(V, E) \longleftrightarrow \overline{G}(\overline{V}, \overline{E})$

More . . . [GDBCK21]

Properties of Topological Pyramids

1. Graphs need multi-edge and self-loop, are non-simple!
2. dual graph pyramid

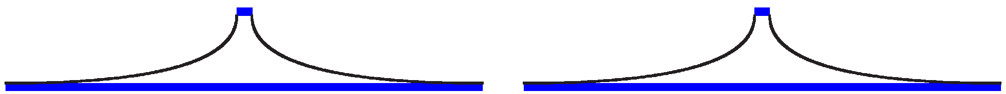
   $\{\oplus, \ominus, \otimes\}$ $\longleftrightarrow$ slope regions

   primal graph $G(V, E)$ $\longleftrightarrow$ dual graph $\overline{G}(\overline{V}, \overline{E})$
3. controlled by application specific
   - selection criteria,
   - reduction functions (could be convolution + activation), and
   - termination criteria
4. pseudo elements characterize topological relations
   2D-ex.:pseudo edge $\longleftrightarrow$ hole, 3D-ex.:pseudo face $\longleftrightarrow$ tunnel,...
5. independence of operations: parallelism see [BBK22]@ICPRAI

Conclusion (1)

- intrinsic 3D (cell-)structure does not change at higher levels
- topological data structures:
  - 2D: planar graphs, combinatorial maps
  - 3D: combinatorial maps, generalized maps
  - nD: generalized maps.
- hierarchies (pyramids) can be built on image, retina, planar graph, combinatorial map, generalized map
- Goal: REDUCE DATA while PRESERVING PROPERTIES

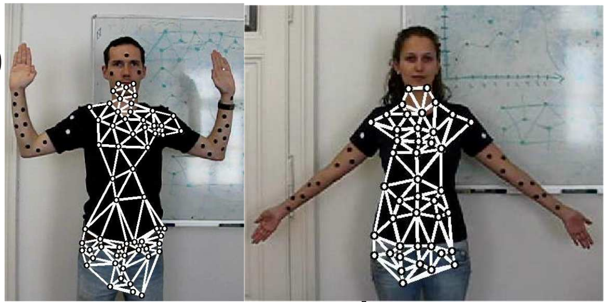
Conclusion (2) and Outlook

- **LBP $\longleftrightarrow$ edge orientation $\longrightarrow$ critical points $\oplus, \ominus, \otimes$ and slopes.**
- **Partitioning into slope regions: not unique**
- **Slope complex partitions continuous surface.**
- **Merge slopes $\Longrightarrow$ dual graph pyramid.**
- **Any hill-climbing inside a slope region reaches the peak!**
- **Complexity: $\mathcal{O}(\log(\textbf{diameter (slope)}))$ following parent links.**
- **DCNN $\longleftrightarrow$**  ??

Contributions by ...

2010

1990 2000 2020



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References

[AIK09] Nicole M. Artner, Adrian Ion, and Walter G. Kropatsch. Rigid Part Decomposition in a Graph Pyramid. In Jan Olof Eklundh Eduardo Bayro-Corrochano, editor, *The 14th International Congress on Pattern Recognition, CIARP 2009*, volume 5856 of *Lecture Notes in Computer Science*, pages 758–765, Berlin Heidelberg, November 2009. Springer-Verlag.

[BBK22] Majid Banaeyan, Darshan Batavia, and Walter G. Kropatsch. Removing Redundancies in Binary Images. In *2nd Iinternational Conference on Intelligent Systems & Pattern Recognition*. Springer LNCS, 2022. doi: 10.1109/IPRIA53572.2021.9483533.

[BGK21] Darshan Batavia, Rocio Gonzalez-Diaz, and Walter G. Kropatsch. Image = Structure + Few Colors. In Andrea Torsello, Luca Rossi, Marcello Pelillo, Battista Biggio, and Antonio Robles-Kelly, editors, *S+SSPR 2020*, volume LNCS 12644 of *Lecture Notes in Computer Science*, pages 365–376. Springer Nature, January 2021.

[BHK19] Darshan Batavia, Jiří Hladůvka, and Walter G. Kropatsch. Partitioning 2D Images into Prototypes of Slope Region. In Mario Vento and Gennaro Percannella, editors, *Proc. 18th Intl. Conference on Computer Analysis of Images and Patterns*, volume LNCS 11678 of *Lecture Notes in Computer Science*, pages pp. 363—-374, Salerno, I, September 2019. Springer Nature Switzerland AG.

[BK98] Mark Burge and Walter G. Kropatsch. Contracting Line Images using Run Graphs. In M. Gengler, M. Prinz, and E. Schuster, editors, *Pattern Recognition and Medical Computer Vision 1998, 22nd ÖAGM Workshop*, pages 235–244. OCG-Schriftenreihe, Österr. Arbeitsgemeinschaft für Mustererkennung, R. Oldenburg, 1998. Band 106.

[BK99] Mark Burge and Walter G. Kropatsch. A Minimal Line Property Preserving Representation of Line Images. *Computing, Devoted Issue on Image Processing*, 62:pp. 355–368, 1999.

[BK21] Majid Banaeyan and Walter G. Kropatsch. Pyramidal Connnected Component Labeling by Irregular Pyramid. In *5th Iinternational Conference on Pattern Recognition and Image Analysis*. IEEE, 2021.

[CGK15] Martin Cerman, Rocio Gonzalez-Diaz, and Walter G. Kropatsch. Lbp and irregular graph pyramids. In Nicolai Petkov and George Azzopardi , editors, *Computer Analysis of Images and Patterns*, volume LNCS 9256 of *Lecture Notes in Computer Science*, Malta, September 2015. Springer, Cham.

[CJGK16] Martin Cerman, Ines Janusch, Rocio Gonzalez-Diaz, and Walter G. Kropatsch. Topology-based image segmentation using LBP pyramids. *Machine Vision and Applications*, 27(8):1161–1174, 2016.

[GDBCK21] Rocio Gonzalez-Diaz, Darshan Batavia, Rocio M Casablanca, and Walter G Kropatsch. Characterizing slope regions. *Journal of Combinatorial Optimization*, pages 1–20, 2021.

[GEK99] Roland Glantz, Roman Englert, and Walter G. Kropatsch. Contracting distance maps of pores to pore networks. In Norbert (ed.) Brändle, editor, *Computer Vision - CVWW'99, Proceedings of the Computer Vision Winter Workshop*, pages 112–121, Wien, Austria, 1999. PRIP TU Wien.

[GPK02] Roland Glantz, Marcello Pelillo, and Walter G. Kropatsch. Matching Hierarchies of Segmentations. In Horst Wildenauer and Walter G. Kropatsch, editors, *Computer Vision - CVWW'02, Computer Vision Winter Workshop*, pages pp. 149–158, Wien, Austria, February 2002. PRIP, TU Wien.

[GPK04] Roland Glantz, Marcello Pellilo, and Walter G. Kropatsch. Matching Segmentation Hierarchies. *International Journal for Pattern Recognition and Artificial Intelligence*, 18(3):397–424, 2004.

[HIK06] Yll Haxhimusa, Adrian Ion, and Walter G. Kropatsch. Comparing Hierarchies of Segmentations: Humans, Nomalized Cut, and Minimum Spanning Tree. In Frank Lenzen, Otmar Scherzer, and Markus Vincze, editors, *Proceedings of 30th OEAGM Workshop*, pages 95–103, Obergurgl, Austria, 2006. OCG-Schriftenreihe books@ocg.at, Österreichische Computer Gesellschaft. Band 209.

[HK03] Yll Haxhimusa and Walter G. Kropatsch. Hierarchy of Partitions with Dual Graph Contraction. In E. Michaelis and G. Krell, editors, *DAGM 2003, 25th DAGM Symposium*, volume 2781 of *Lecture Notes in Computer Science*, pages 338–345, Magdeburg, Germany, September 2003. Springer, Berlin Heidelberg.

[HK04] Yll Haxhimusa and Walter G. Kropatsch. Segmentation Graph Hierarchies. In Ana Fred, Terry Caelli, Robert P.W. Duin, Aurelio Campilho, and Dick de Ridder, editors, *Structural, Syntactic, and Statistical Pattern Recognition, Joint IAPR International Workshops on SSPR 2004 and SPR 2004*, volume LNCS 3138 of *Lecture Notes in Computer Science*, pages 343–351, Lisbon, Portugal, August 2004. Springer, Berlin Heidelberg, New York.

[HMK04] Allan Hanbury, Jocelyn Marchadier, and Walter G. Kropatsch. The Redundancy Pyramid and its Application to Image Segmentation. In Wilhelm Burger and Josef Scharinger, editors, *Digital Imaging in Media and Education, 28th ÖAGM*

Workshop, pages 157–164. OCG-Schriftenreihe, Österr. Arbeitsgemeinschaft für Mustererkennung, R. Oldenburg, 2004. Band 179.

[HR84] T.-H. Hong and Azriel Rosenfeld. Compact region extraction using weighted pixel linking in a pyramids. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. PAMI-6(2):pp.222–229, 1984.

[JM92] Jean-Michel Jolion and Annick Montanvert. The adaptive pyramid, a framework for 2D image analysis. *Computer Vision, Graphics, and Image Processing: Image Understanding*, 55(3):pp.339–348, May 1992.

[KB96a] Walter G. Kropatsch and Souheil BenYacoub. A general pyramid segmentation algorithm. In Robert Melter, Angela Y. Wu, and Longin Latecki, editors, *Vision Geometry V, Intl. Symposium on Optical Sciences, Engineering, and Instrumentation*, volume 2826, pages 216–224. SPIE, 1996.

[KB96b] Walter G. Kropatsch and Souheil BenYacoub. Universal Segmentation with P*IRR*amids. In Axel Pinz, editor, *Pattern Recognition 1996, Proc. of 20th ÖAGM Workshop*, pages 171–182. OCG-Schriftenreihe, Österr. Arbeitsgemeinschaft für Mustererkennung, R. Oldenburg, 1996. Band 90.

[KB98] Walter G. Kropatsch and Mark Burge. Minimizing the Topological Structure of Line Images. In Adnan Amin, Dov Dori, Pavel Pudil, and Herbert Freeman, editors, *Advances in Pattern Recognition, Joint IAPR International Workshops SSPR'98 and SPR'98*, volume Vol. 1451 of *Lecture Notes in Computer Science*, pages 149–158, Sydney, Australia, August 1998. Springer, Berlin Heidelberg, New York.

[KBI98] Walter G. Kropatsch, Mark Burge, and Heinz L. Idl. Dual Graph Contraction for Run Graphs. In Aleš Leonardis and Franc (eds.) Solina, editors, *Computer Vision - CVWW'98, Proceedings of the Computer Vision Winter Workshop*, pages pp. 75–86, Ljubljana, 1998. IEEE Slovenia Section.

[KCBG19a] Walter G. Kropatsch, Rocio M. Casablanca, Darshan Batavia, and Rocio Gonzalez-Diaz. Computing and Reducing Slope Complexes. In Rebeca Marfil, Mariletty Calderon, Fernando Diaz del Rio, Pedro Real, and Antonio Banderas, editors, *Proceedings 7th intl. Workshop on Computational Topology in Image Context*, volume LNCS 11382 of *Lecture Notes in Computer Science*, pages 12—-25, Malaga, Spain, January 2019. Springer, Berlin Heidelberg.

[KCBG19b] Walter G. Kropatsch, Rocio M. Casablanca, Darshan Batavia, and Rocio Gonzalez-Diaz. On the Space Between Critical Points. In Michel Couprie, Jean Cousty, Yukiko Kenmochi, and Nabil Mustafa, editors, *Discrete Geometry for Computer*

Imagery, volume LNCS 11414 of *Lecture Notes in Computer Science*, pages 115–126, Marne-la-Vallée, F, March 2019. Springer, Berlin Heidelberg.

[KH04a] Walter G. Kropatsch and Yll Haxhimusa. Grouping and Segmentation in a Hierarchy of Graphs. In Charles A. Bouman and Eric L. Miller, editors, *Computational Imaging II*, volume 5299, pages 193–204, Bellingham, WA, January 2004. SPIE.

[KH04b] Walter G. Kropatsch and Yll Haxhimusa. Hierarchical Grouping of Non-connected Structures. In Wilhelm Burger and Josef Scharinger, editors, *Digital Imaging in Media and Education, 28th ÖAGM Workshop*, pages 165–172. OCG-Schriftenreihe, Österr. Arbeitsgemeinschaft für Mustererkennung, R. Oldenburg, 2004. Band 179.

[KM95] Walter G. Kropatsch and Herwig Macho. Finding the structure of connected components using dual irregular pyramids. In *Cinquième Colloque DGCI*, pages 147–158. LLAIC1, Université d'Auvergne, ISBN 2-87663-040-0, September 1995.

[Koe84] Jan J. Koenderink. The structure of images. *Biological Cybernetics*, 50:363–370, 1984.

[Kro95] Walter G. Kropatsch. Building Irregular Pyramids by Dual Graph Contraction. *IEE-Proc. Vision, Image and Signal Processing*, Vol. 142(No. 6):pp. 366–374, December 1995.

[Kro96] Walter G. Kropatsch. Properties of pyramidal representations. *Computing, Supplementum: Theoretical Foundations of Computer Vision*, No. 11:pp. 99–111, 1996.

[Kro97] Walter G. Kropatsch. Property Preserving Hierarchical Graph Transformations. In Carlo Arcelli, Luigi P. Cordella, and Gabriella Sanniti di Baja, editors, *Advances in Visual Form Analysis*, pages 340–349. World Scientific Publishing Company, 1997.

[Kro02] Walter G. Kropatsch. Abstraction Pyramids on Discrete Representations. In Achille Braquelaire, Jacques-Olivier Lachaud, and Anne Vialard, editors, *Discrete Geometry for Computer Imagery, 10th DGCI*, volume Vol. 2301 of *Lecture Notes in Computer Science*, pages 1–21, Bordeaux, France, 2002. Springer, Berlin, Heidelberg, New York.

[Mee89] Peter Meer. Stochastic image pyramids. *Computer Vision, Graphics, and Image Processing*, Vol. 45(No. 3):pp.269–294, March 1989.

[MFTM01] David R. Martin, Charless Fowlkes, Doron Tal, and Jitendra Malik. A Database of Human Segmented Natural Images and its Application to Evaluating Segmentation Algorithms and Measuring Ecological Statistics. In *8th International Conference on Computer Vision, ICCV 2001*, volume 2, pages 416–423, 2001.

[MK95] Herwig Macho and Walter G. Kropatsch. Finding Connected Components with Dual Irregular Pyramids. In Franc Solina and Walter G. Kropatsch, editors, *Visual Modules, Proc. of 19th ÖAGM and 1st SDVR Workshop*, pages 313–321. OCG-Schriftenreihe, Österr. Arbeitsgemeinschaft für Mustererkennung, R. Oldenburg, 1995. Band 81.

[MKH04] Jocelyn Marchadier, Walter G. Kropatsch, and Allan Hanbury. The Redundancy Pyramid and its Application to Segmentation on an Image Sequence. In Carl Edward Rasmussen, Heinrich H. Bülthoff, Martin A. Giese, and Bernhard Schölkopf, editors, *DAGM Symposium 2004*, volume LNCS 3175 of *Lecture Notes in Computer Science*, pages 432–439, Tübingen, Germany, September 2004. Springer, Berlin Heidelberg.

[OPH96] Timo Ojala, Matti Pietikäinen, and David Harwood. A comparative study of texture measures with classification based on featured distributions. *Pattern recognition*, 29(1):51–59, 1996.

[Piz22] Zygmunt Pizlo. *Solving Solving: Cognitive Mechanisms and Formal Models*. Cambridge University Press, Cambridge, August 2022.

[PKJ98] Jean-Gerard Paillоncy, Walter G. Kropatsch, and Jean-Michel Jolion. Object Matching on Irregular Pyramid. In Anil K. Jain, Svetha Venkatesh, and Brian C. Lovell, editors, *14th International Conference on Pattern Recognition*, volume II, pages 1721–1723. IEEE Comp.Soc., 1998.

[TK14] Fuensanta Torres and Walter G. Kropatsch. Canonical Encoding of the Combinatorial Pyramid. In Zuzana Kúkelová and Jan Heller, editors, *Proceedings of the 19th Computer Vision Winter Workshop 2014*, pages 118–125, Křtiny, CZ, February 2014. ISBN: 978-80-260-5641-6.

[Uhr86] Leonard Uhr. Parallel, hierarchical software/hardware pyramid architectures. In Virginio Cantoni and Stefano Levialdi, editors, *Pyramidal Systems for Image Processing and Computer Vision*, volume F25 of *NATO ASI Series*, pages 1–20. Springer-Verlag Berlin, Heidelberg, 1986.