Technical Report

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IRREGULAR PYRAMIDS

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Abstract

A pyramid is a stack of images with exponentially decreasing resolutions. Many image processing algorithms run on this hierarchical structure in $O(\log n)$ parallel processing steps where n is a side of the input image. Perturbations in the structure may disturb the originally regular neighborhood relations and also the stability of the results. On the other hand, biological vision is based on piecewise regular patches in the retina, e.g. of a monkey or a human. P. Meer's stochastic pyramid is such an irregular structure. The parallel generation of the structure is governed by two "decimation rules" that also characterize a maximal independent set on the neighborhood graph of the image pixels. In general, the number of neighbors in the decimated graph may increase. It is shown that the decimation $G'(V', E')$ of any neighborhood-graph $G(V, E)$ preserves the degrees in the corresponding dual graphs. However the dual of G' is not always a "good" decimation of the dual of G. Investigating in parallel dual decimations of regular graphs, one finds unique solutions that have interesting properties for image pyramids. Besides the above theoretical motivation for irregular structures, we can find similar strutures in the retinas of monkeys (and also of humans). This report combines the results published in the proceedings of two conferences [10, 9] with a few additional findings.

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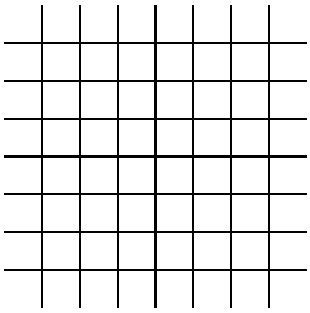
1 Introduction

Current vision systems are based on two different types of sensing structures: digital cameras produce regular square grids, whereas humans or monkeys build their 'images' in the retina, a sensing structure which shows a certain degree of irregularity (Fig. 1). A closer



(a) (b) æ

Figure 1: Section of (a) monkey's retina and of (b) digital image

look shows that the hexagonal impression of the biological structure also contains cells with more and with fewer than six neighbors. Although technical systems claim to 'see', they do not reach the performance of humans or even of monkeys by order of magnitude [22]. In this paper we concentrate on these structural differences and we investigate in the theoretical background of also allowing a certain degree of irregularity in technical systems.

Mathematically, an image is defined by a function $f(x, y) : \mathcal{R}^2 \mapsto \mathcal{R}$ of two variables in the image plane [20, Vol.I, chapter 1]. A digital image is formed by a *sampling process* that extracts a discrete set of numbers (e.g., gray values). This set has two possible interpretations: either a) as measurements at discrete points in the image plane; or b) as a collection of 'cells' that partition the image plane. Interpretation (b) describes the contents of the respective image in terms of the regions that are covered by the respective cells.

The *sampling structure* of a digital image is defined by the distribution of the sampling points in the case (a), and by size, shape, and topology of the partition cells in the case (b). Let us describe the structure of a digital image by the neighborhood-graph, $G(V, E)$, with vertices V and edges E.Two vertices, $p, q \in V$ are connected in G, $(p, q) \in E$, if they are *neighbors* in the structure. We shall denote the neighborhood of a vertex $p \in V$ by $\Gamma(p) := \{p\} \cup \{q \in V | (p, q) \in E\}$. A structure is called *regular* if a well-defined neighbor-relation holds for all vertices (except on the boundary).

In geometry, a pyramid is defined as a polyhedron with a polygonal *base* and (triangular) faces meeting in *a common vertex*. In image analysis, an *image pyramid* is a stack of digital images (the levels V_i of the pyramid) with local level-to-level relations. A regular pyramid structure can be defined by three parameters [8]: (1) pixels of the **reduction window** are input to (2) a **reduction function** , that outputs one pixel of the next level. It has to be computed for every cell of the next level. (3) The **reduction factor** determines the rate by which the number of cells decrease from level to level.

In a regular pyramid, the reduced resolution representations can be distorted when the input is shifted [21]. Bister et al [2] show that most classical pyramids, which are regular and consequently rigid structures, have problems in segmenting an image, e.g., when the input image is shifted, rotated, or differently scaled. This shift-dependence problem will appear whatever the neighborhoods which are used to built the pyramid (that is non-overlapping or overlapping neighborhoods). It is a forseeable phenomenon since an input picture is a random process (position and shape of the objects) that we try to match on a rigid grid at different resolutions (the rigid pyramid). Trials to solve these problems, such as iterative modifications of parent-child links or of link weights, remain partial solutions [6][18].The main idea of irregular pyramids is to allow the structure to be flexible enough to match with the input data, such that the image contents controls the aggregation process.

The first step in image analysis, segmentation, produces a set of (connected) regions that correspond to objects in the image. The adjacency relations among these regions are described by the region-adjacency-graph (RAG), which is in general not regular.

How to build a structure that relates the digital input image with a generalized graph will be explained in section 2. This procedure can be applied on both binary pictures and grayscale pictures (section 3). In the case of binary pictures, it performs *connected components extraction* in $\mathcal{O}(\log(\text{largest component-diameter}))$, and provides at the top of the pyramid the adjacency graph between the components (RAG). In the case of grayscale pictures, it performs *image segmentation*; at the top-level, each object is represented by one vertex and the neighborhood relations between them provide the region adjacency graph.

Regular pyramids show several properties that contribute to their efficiency. Irregular pyramids have different properties (section 4). In addition we will focus on the following property of all regular pyramids (section 5): The structural parameters are constant, especially the number of neighbors (e.g., the degree of the neighborhood-graph) remains the same at all pyramid levels. In general, irregular pyramids do not have this property. A weaker property requires the degree of the neighborhood-graphs to remain bounded by a pre-specified value. If we consider the corresponding dual graphs, we find that the degrees of the dual vertices do not increase. Based on this observation, parallel dual decimations are found for new types of regular pyramids (section 6).

2 Bottom-up Construction of Pyramids

There are two main ways to build a pyramid from a digital image in its base: a) by graph contraction or b) by decimation.

2.1 Parallel graph contraction

Parallel graph contraction [19] represents every level of the pyramid by the neighborhood-graph. The nodes of the graph are the pyramidal cells. Higher level graphs are created by successively merging a certain number of nodes in the lower level graph.

Rosenfeld [19] explores different sorts of cellular hierarchies in a general theory. His focus is on methods that allow a parallel generation of the structure. He states the problem to find a parallel contraction scheme such that **the degree of the contracted graph remains bounded**. This important property would, for example, allow to simulate large networks of processors by smaller ones. For some special, regular cases parallel contraction is possible: hypercube, k-dimensional array, hexagonal array. However the degree of triangular array increases.

2.2 Decimation

Decimation divides the cells of a pyramid level V_i into two categories: cells that **survive** the decimation process and form the cells of level V_{i+1}; and cells $v \in V_i$ that do not appear at higher levels, $v \notin V_{i+1}, v \notin V_{i+2}, \ldots$ (**non-surviving cells**). In a rigid pyramid it is known a priori which cells survive. Relaxing this rigidity, the selection of surviving cells depends on the input, but it will be decided by a bottom-up process (starting from the input picture). Non-surviving cells ($v \in V_i - V_{i+1}$) have to give a contribution to define the level i. This contribution is provided by the construction of son-father links, that is, assignment of non-survivors to survivors. The resulting structure is a multiresolution graph. At a given level i, a vertex $v_i \in V_i$ has some **children** on level $i-1$, a **father** on the level $i+1$ above, $FV_i(v_i) \in V_{i+1}$, and some **neighbors** on level i, $\Gamma_i(v_i) \subset V_i$. All children of a vertex form its **receptive field** $RF_i(v_i) \subset V_{i-1}$.

To build an irregular pyramid by decimation, three sub-problems have to be solved: (1) how to select the survivors, $V_{i+1} \subset V_i$; (2) how to assign the survivors (son-father links), RF_{i+1}, FV_i; (3) how to define the neighbors of a survivor, Γ_{i+1}. Then the same process will be applied to build the whole pyramid level after level, starting from the bottom. Peter Meer defined such a process in which problems 1 and 2 were solved by a stochastic process following two rules [12, 13]: a) two neighbors on level i cannot survive both; b) a non-survivor must be a neighbor of a survivor.

These two rules are equivalent to saying that the vertices V_{i+1} of the graph $G_{i+1}(V_{i+1}, E_{i+1})$ on level $i+1$ define a Maximum Independent (vertex) Set (MIS) of the graph $G_i(V_i, E_i)$ [3, chapter 3], [11].The main interest of the MIS is that it can be computed in parallel for a graph, and using only local operations, what is a key point in pyramid concepts. The reduction in size of the irregular pyramid is provided by this MIS extraction. The process developed is different from other parallel processes in the sense that it will choose the survivors without errors (that is, without having to remove some of them in case one of the two rules is not respected).

Algorithm: Stochastic decimation

1. Assign uniformly distributed random numbers to all cells $v_i \in V_i$: $g(v_i) \in (0, M) \subset \mathcal{R}$.
2. Select local maxima as surviving cells: $v_i \in V_{i+1}$ if $g(v_i) > g(v_j)$ for all $v_j \in \Gamma_i(v_i)$.

3. Fill holes: a) set g(v):=0 for all $v \in V_i : \Gamma_i(v) \cap V_{i+1} \neq \emptyset$; b) repeat step 2 if not all $g(v_i) = 0$.

4. Every non-surviving cell $v_i \in V_i - V_{i+1}$ selects a father $FV_i(v_i) \in V_{i+1} \cap \Gamma_i(v_i)$. This construction also defines the receptive fields: $RF_{i+1}(v_{i+1}) := \{v_i \in V | FV_i(v_i) = v_{i+1}\}$.

5. A surviving vertex $p_{i+1} \in V_{i+1}$ becomes a neighbor of vertex $q_{i+1} \in V_{i+1}$ at level $i+1$, $p_{i+1} \in \Gamma_{i+1}(q_{i+1})$, if there exists a vertex $p_i \in RF_{i+1}(p_{i+1})$ that has a neighbor in $RF_{i+1}(q_{i+1})$; i.e. $\Gamma_i(p_i) \cap RF_{i+1}(q_{i+1}) \neq \emptyset$.

6. Set $i := i + 1$ and repeat steps 1 - 6 until $card(V_i) = 1$ [1].

3 Segmentation by Decimation

Pyramids are made to work on an input picture. From this stochastic process of building an irregular structure, a process guided by the data can now be defined. We would like the reduction to occur in each object, whatever its location in the picture. When we reach the apex of the pyramid, each object should be represented by one survivor. This will be done as we explain below firstly on binary images and secondly on grayscale pictures. (For more detail, see [16, 17].)

3.1 Connected components extraction

When a vertex has to decide if it will survive or not at the next level, it looks at its neighbors; rather than checking all its neighbors, a vertex will only check its neighbors which share its color (black or white). This is equivalent to computing a MIS inside each connected component, but it is performed just by adapting the previous process with some local constraints. In the second step, a non-survivor chooses a neighbor surviving which shares its color; due to the constraints, such a vertex always exists. The third step doesn't have to change. So the reduction is performed inside each connected component, but involving only local and parallel operations. Each individual hierarchy, down projecting a vertex from the top to the bottom provides a connected component of the initial picture (Fig. 2).

3.2 Grayscale image segmentation

The technique described above is based on a binary decision. Two neighbor vertices are put in the same similarity class if they have the same color.

Starting with a grayscale picture at the bottom, the decision cannot be so clear to take. We have to analyse the neighborhood of a vertex more precisely. We will not give here too many details, the important point is that each vertex will analyse its neighborhood to sort its neighbors into bad, not very similar and similar classes. And each vertex will take the

[1] The cardinality $card\{S\}$ of a set S is the number of elements in S.

Figure 2: Two segmented binary images

decision to survice due to a local process. The non-surviving vertices are assigned to their most similar surviving neighbor. The survivors have to carry information related to their receptive fields, that is the average gray level. The changes compared to the binary case are that now the similarities computed between the vertices might not be symmetric, and that connected components are no more well-defined. Indeed we will reach the apex of the irregular pyramid with a segmented picture, which is not a priori known.

Due to the stochastic contribution in the MIS computation, receptive fields down projected on the bottom (that is the segmentation of the initial picture) might change when running several times the process. But the main components are always well extracted. More than that, this defines a way to measure the quality of each extracted region by confirming its presence in the final segmentation [14].

4 Some Properties of Irregular Pyramids

Properties that are trivial for regular pyramids are not obvious in irregular structures, some are even not present. In this section, we study the geometric ground distance between neighbor cells at a given pyramid level. This is related to the sampling (Nyquist) distance corresponding to the chosen level.

In any regular $n \times n/f$ pyramid the distance between neighbors at level i is $f^{i/2}$. We will show that in 1D irregular pyramids, 2^i and 3^i are the bounds for 1D neighbors at level i. Note that these bounds depend on the level i but not the structure of the pyramid. We will derive a decimation in 2D where geometrical distances between neighbors do not necessarily increase at the decimated levels.

Another important property is the decimation ratio (this is equivalent to the reduction factor for regular pyramids) between adjacent pyramid levels. For stochastic pyramids Meer [12] has found an average decimation ratio greater than 4 in his experiments. We have found decimations that achieve a decimation ratio of 2. Even ratios as low as 4/3 are possible and can be realized for certain graph structures.

4.1 Distances in Irregular 1D-Pyramids

On a 1D regular sampling grid V_0 we define a distance d by $\forall x \in \mathcal{L}_0 : d(x,x) = 0$, and $d(x,y)$ is the minimum number of steps to go from x to y such that all cells on that path are neighbors of each other. Note that such a path is unique in 1D. Since all decimated levels V_i are subsets of base level V_0, we can measure the distance between every two cell in any level i by their distance in the base.

Theorem 1 *The distance between two neighbors P_i, Q_i at level V_i of the 1D irregular pyramid is bounded by following inequality:*

$$2^i \leq d(P_i, Q_i) \leq 3^i \tag{1}$$

Proof:

$i = 1$**:** At the first level, P_1 and Q_1 are not allowed to be neighbors in the base, hence $d(P_0, Q_0) > 1$.

The two receptive fields of P_1 and Q_1 must touch each other because P and Q are neighbors at level 1. Therefore there exist cells A and B in the receptive fields $RF(P_1)$ and $RF(Q_1)$ such that A and B are neighbors in the base. Allowing maximum distances the sum is 3: $P_0 - A - B - Q_0$.

$n \rightarrow n+1$**:** Let P_{n+1} and Q_{n+1} be neighbors at level $n+1$. Then they are not allowed to be neighbors at level n, hence there exists a non-surviving cell T_n on the path from P_n to Q_n. Assuming $d(P_n, T_n) \geq 2^n$ and $d(T_n, Q_n) \geq 2^n$ it is concluded that $d(P_n, Q_n) \geq 2^n + 2^n = 2^{n+1}$.

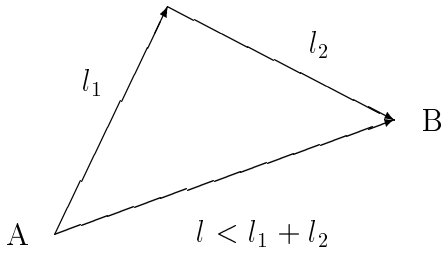
The two receptive fields of P_{n+1} and Q_{n+1} must touch each other because P_{n+1} and Q_{n+1} are neighbors. The maximum number of non-surviving cells at level n between P_n and Q_n is therefore two. Let the configuration be $P_n - A_n - B_n - Q_n$.

$$d(P_{n+1}, Q_{n+1}) \;=\; d(P_n, A_n) + d(A_n, B_n) + d(B_n, Q_n) \leq 3^n + 3^n + 3^n = 3^{n+1} \tag{2}$$

These distances are measured along the base line of the 1D-pyramid in units of the original sampling (in number of chain codes in the chain pyramid [15]). ■

4.2 Distances in Irregular 2D-Pyramids

In 2D we use the Euclidean distance between two cells in the base. This case is not as easy as the 1D case because in 2D the sum of lengths of two straight line segments is not necessarily the distance between the end points:



As in [12], 8-neighborhood is assumed in the base level of the decimation process.

Theorem 2 *The distance between two neighbors P_i, Q_i at level $i > 0$ of the 2D irregular pyramid is bounded by following bounds:*

$$2 \leq d(P_1, Q_1) \leq 3\sqrt{2} \quad (3)$$

$$\sqrt{5} \leq d(P_i, Q_i) \leq 3^i\sqrt{2},\ i > 1 \quad (4)$$

and these bounds can be reached.

Proof:

Upper Bound: The largest distance between two 8 connected neighbors in a square grid is $\sqrt{2}$. As in the 1D case the maximum distance between two surviving cells is reached by the configuration $P - A - B - Q$. Allowing maximum distances between each pair $d(P_1, Q_1) = 3\sqrt{2}$.

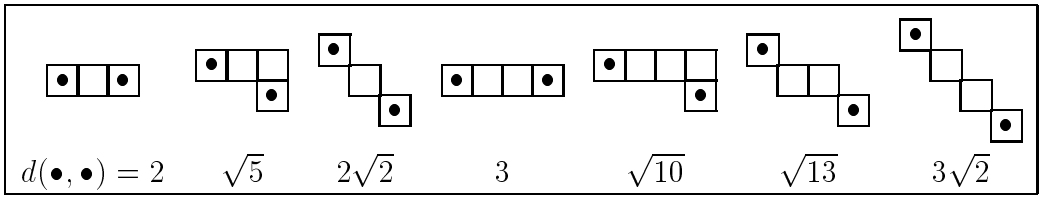
The adjacency graph of level 1 is in general not regular. But the topological configuration $P - A - B - Q$ measured along a straight line gives still an upper bound for the distance between P and Q, hence $d_{max}(P_{i+1}, Q_{i+1}) \geq 3d_{max}(P_i, Q_i)$.

Lower Bound, $i = 1$**:**

n	n	n		
n	P	n	Q	
n	n	n		

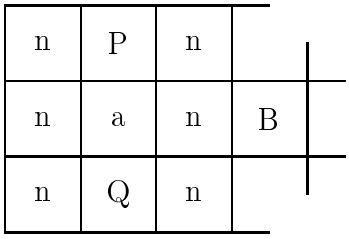
In an 8-connected square grid, P has eight neighbors ('n'). Q is the closest non-neighbor cell to P: $d(P, Q) = 2$.

Lower Bound, $i \geq 2$**:** Two cells P and Q that are not neighbors at level n may both survive to level $n+1$ and, hence, their distance remains the same. At level 1 two cells ($\bullet$) may become neighbors in following configurations of level 0 using 8-connectivity to define the neighborhood (symmetric and rotated versions are not enumerated):



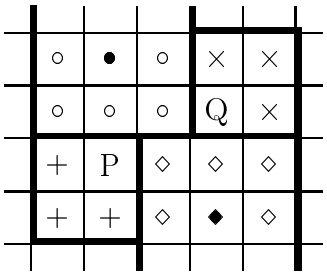
The neighborhood at level 1 depends on the choice of receptive fields $\mathcal{R}$. Two geometrically possible neighbors P_1 and Q_1 may not become neighbors in V_1 if all cells of $\Gamma(P_0) \cap \Gamma(Q_0)$ are assigned to receptive fields other than $RF(P_1)$ and $RF(Q_1)$. Then the two receptive fields, $RF(P_1), RF(Q_1)$, have no common boundary segment and both, P_1 and Q_1, can survive to level 2.

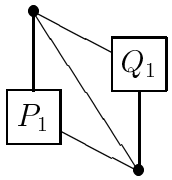
For $d(P_1, Q_1) = 2$, we consider cell a in between P_0 and Q_0 which is in $\Gamma(P_0)$ and in $\Gamma(Q_0)$:



The possible surviving cell closest to a other than P_0 or Q_0 is B. But $a \notin \Gamma(B)$ so that a may be only assigned to either $RF(P_1)$ or to $RF(Q_1)$. Therefore $P_1 \in \Gamma(Q_1)$ cannot survive to level 2 if Q_1 survives.

For $d(P_1, Q_1) = \sqrt{5}$, P_1 need not become a neighbor of Q_1 if $RF(P_1)$ and $RF(Q_1)$ are chosen as follows (cells $P, Q, \bullet$ survive to level 1):





neighbors at level 1

In this case, both P_1 and Q_1 survive to level 2 and higher levels as long as the common neighbors can be assigned to receptive fields other than $RF(P_i)$ and $RF(Q_i)$. Therefore the minimum distance remains $\sqrt{5}$ and does not increase in such cases.

■

4.3 Decimation Ratios in Irregular 2D-Pyramids

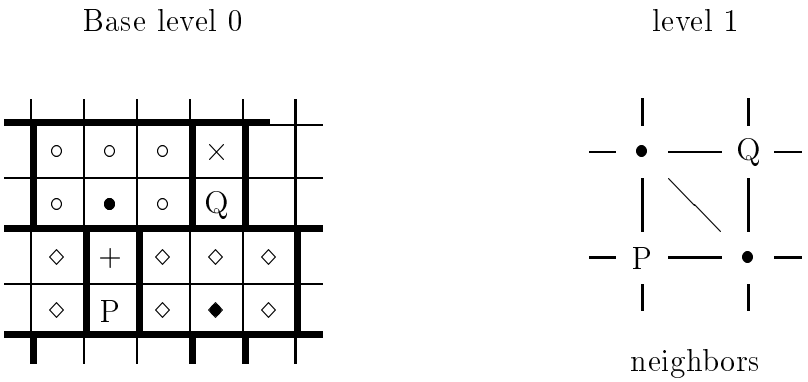
We define the decimation ratio $\mathcal{D}_i$ between two levels i and $i+1$ as follows:

$$\mathcal{D}_i = \frac{card(V_i)}{card(V_{i+1})} \tag{5}$$

Since the minimum distance does not necessarily increase during the decimation process (Theorem 2), and since, on the other hand, the decimation eliminates a certain number of cells, we search for bounds of the decimation ratio. Meer [12] found an average decimation ratio greater than 4. We found examples that achieve lower ratios.

Theorem 3 *A decimation ratio of $\mathcal{D}_i = 2$ at levels i greater than 0 is possible in the 2D irregular pyramid even if 8-connectivity is used in the base level.*

Proof:



P and Q are not neighbors at level 1, hence they can both survive to level 2. The decimation ratio for this regular grid V_1 is $\mathcal{D}_0 = 4$. For a larger number of cells the neighbor relations take the following (regular) structure (Figure 3):

Let the degree of a cell be the number of its neighbors. Then there is an equal number of cells with degree 4 (•) and with degree 8 (◯) in this structure (Fig. 3a). The average degree is 6. The cells of degree 4 are all non-adjacent and can be chosen to survive the decimation. Hence a possible decimation ratio is $\mathcal{D}_1 = 2$. This can be achieved if the receptive fields are chosen as indicated in Fig. 3b. The corresponding neighborhood relations at level 2 (Fig. 3c) have the same structure as in level 1. Hence this decimation process can be repeated while maintaining a decimation ratio of 2. The geometric relation is a 45^0 rotation between two successive levels and the number of cells at the higher level is 1/2 of the level directly below. Note that this regular pyramid structure is equivalent to the $2 \times 2/2$ pyramids used by Crowley [4] and later by Kropatsch [7]. ■

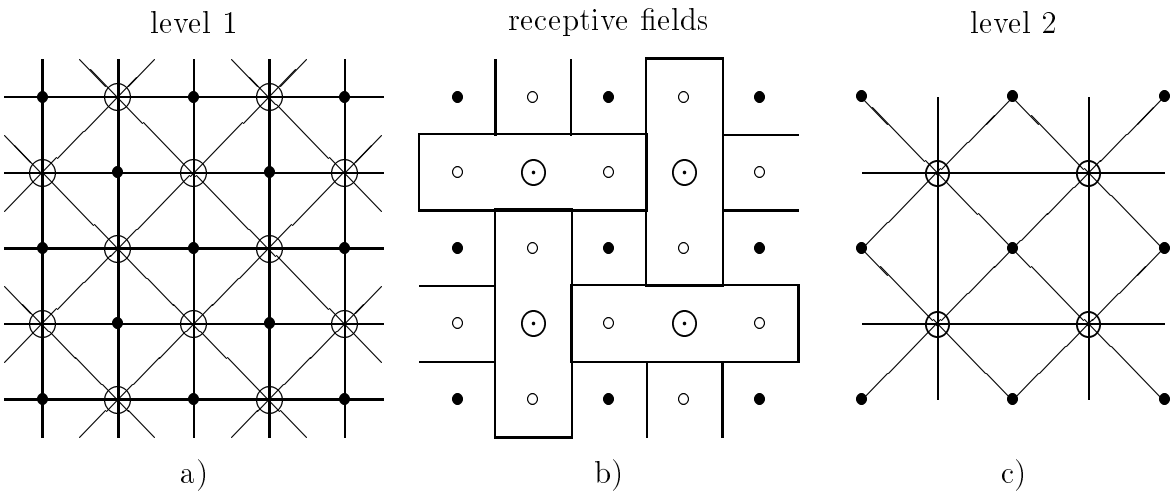
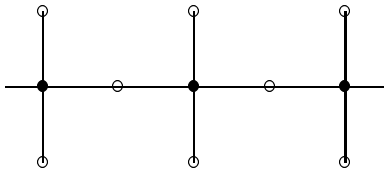


Figure 3: Square grid with decimation ratio 2

On a 2D-square lattice, two is the smallest decimation ratio that fulfills the first decimation rule, e.g. two neighbors should not both survive. Hence two surviving cells must be separated in the level below by at least one non-surviving cell. This must be true for a row of the square lattice: . But it also implies some of the non-surviving cells in the rows above and below:



One surviving cell in a row needs a non-surviving cell in the adjacent row. Since the number of surviving cells equals the number of non-surviving cells in the first row and if all the non-determined cells survive the row-by-row propagation, half of the cells will survive. Hence the smallest decimation ratio on a square lattice is two. Note that additional neighborhood relations are not explicitely excluded by this construction and that the diagonal neighbors of the minimal graph still obey the decimation rules. Therefore a decimation ratio of 2 can be realized.

4.3.1 Decimation Ratios less than Two

So far we only considered neighborhood structures that are defined on square grids. But higher levels can become general graph structures. And for those even lower decimation ratios can be achieved. Since we do not derive the proposed graph structure from a square grid it is not clear if this would be possible in all cases.

We observe that the decimation ratio is lower if the degree of the surviving cells is small. We further assume that there is at most one arc between any pair of nodes in the graph, we exclude multiple neighbors. Since degree 1 does not seem reasonable we consider a graph where all surviving cells have degree 2 at level i. Cells with higher degrees would produce greater decimation ratios. This addresses the problem of preserving the maximum degree in parallel graph contraction. Rosenfeld [19] describes solutions based on algebraic constraints but a general solution is not available.

The neighborhood graph is assumed to be planar. Simply connected planar graphs can be decomposed into primitive cyclic paths. We construct our neighborhood graph by putting together such primitive cycles that consist of alternating sequences of surviving and non-surviving cells (Fig. 4a). If these cycles are joint at non-surviving cells, all intermediate surviving cells have degree 2. The primitive cycles may consist of any number of surviving cells (e.g. 4 in Fig. 4b or 6 in Fig. 4c). Cycles with different numbers of cells may also be combined.

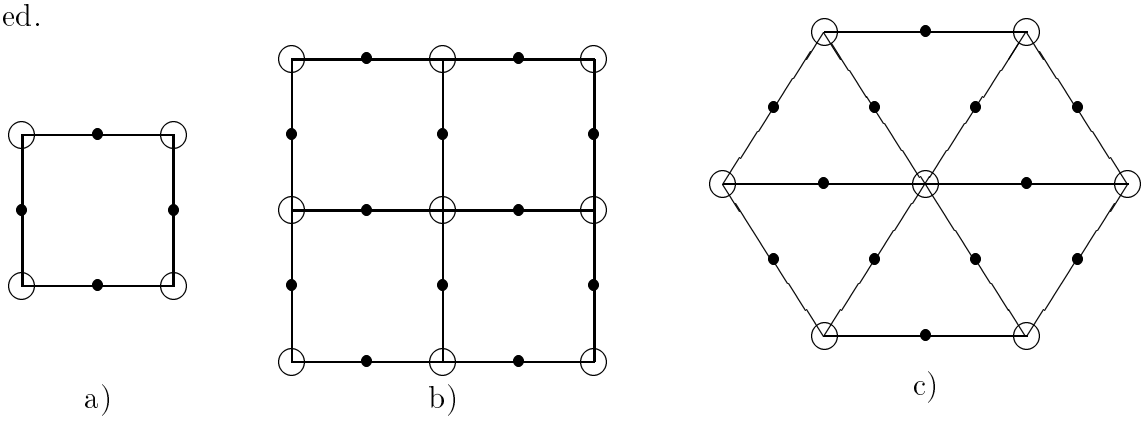


Figure 4: Graph construction by cycles

4.3.2 Triangular Network

We calculate the decimation ratio for the case of Fig. 4c. The non-surviving cells of this structure form a (regular) hexagonal network, the dual of which is a regular triangular network. Every surviving cell is located on one triangle side which we call *edge* of the neighborhood graph. We now derive formulas that relate the number of triangles F ('faces') with the number of vertices P ('points') and the number of edges E of the corresponding graph structure. Since we consider finite structures we differentiate between edges and vertices on the boundary, BE and BP, and inside the structure, IE and IP. The boundary is a closed cycle on which vertices and edges alternate:

$$card(BP) = card(BE) \tag{6}$$

The Euler number for the planar graph can be expressed as follows:

$$card(IP) - card(IE) + card(F) = 1 \tag{7}$$

Every triangle is formed by three edges, every inner edge has one triangle on each side, and a boundary edge delimits only one triangle, the other side faces the background (which we do not count here):

$$3\,card(F) = 2\,card(IE) + card(BE) \tag{8}$$

The number of cells at level V_i is therefore the following sum:

$$\begin{aligned} card(V_i) = & \quad card(\circ) & +card(\bullet) = & \quad (9)\\ = & \quad card(IP) + card(BP) & +card(IE) + card(BE) & \quad (10) \end{aligned}$$

At V_{i+1} only the edges carry a surviving cell ($\bullet$):

$$card(V_{i+1}) = card(\bullet) = IE + BE \tag{11}$$

Taking $card(F)$ and $card(BP)$ as parameters of a given network, the decimation ratio can be expressed in terms of these two parameters:

$$\mathcal{D}_i = 1 + \frac{1 + \frac{card(F)+card(BP)}{2}}{card(F) + \frac{card(F)+card(BP)}{2}} = 2 - \frac{2}{3} + \frac{2}{3}\,\frac{3 + card(BP)}{3\,card(F) + card(BP)} \tag{12}$$

This decimation ratio is less than 2, as can be verified easily, but also greater than $\frac{4}{3}$ (because $card(BP) \geq 0$ always). For large networks this lower bound is closely approached. Note also, that the regularity of the network is not a necessary requirement.

4.3.3 A lower limit

In the previous section the network generating cycle was formed by three edges. If we build networks with cycles of d edges and vertices (like $d = 4$ in Fig. 4b) only equation (8) has to be modified:

$$d \cdot card(F) = 2\,card(IE) + card(BE) \tag{13}$$

Then the decimation ratio becomes

$$\mathcal{D}_i = 1 + \frac{(d-2)\,card(F) + card(BP) + 2}{d \cdot card(F) + card(BP)} = 2 - \frac{2}{d} + \frac{2}{d} \cdot \frac{card(BP) + d}{d \cdot card(F) + card(BP)} \tag{14}$$

For large values of $card(F)$, $\mathcal{D}_i$ approaches $2 - \frac{2}{d}$ which becomes smallest for small d. Since $d = 3$ is the smallest, $\boxed{\mathcal{D}_i \geq \frac{4}{3}}$ for all other combinations as well.

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Figure 5: Dual graphs of monkey's retina and its decimation

5 Decimating image graphs

We apply the above decimation procedure to the data points obtained from a monkey's retina and we derive a stochastically decimated retina (Fig. 5). We observe that both are triangulations. We therefore investigate in this section whether there is a general principle behind this observation.

Let graph $G(V, E)$ describe the structure of a digital image and let $\overline{G}(F, \overline{E})$ be its **dual graph** consisting of **faces** $f \in F$ which are surrounded by edges $e \in E$ of G. Two faces $f_1, f_2 \in F$ are connected in $\overline{G}$, $(f_1, f_2) \in \overline{E}$, if they are separated in G by an edge $e = (v_1, v_2) \in E$. In our monkey's retina, Fig. 1 a) corresponds to $G(V, E)$ and Fig. 5 a) corresponds to $\overline{G}(f, \overline{E})$.

We start with some common definitions from graph theory. A **cycle** is an ordered sequence of connected vertices, $C = (v_0, v_1, \ldots, v_n); v_i \in V, i = 0, \ldots, n$, $(v_i, v_{i+1}) \in E; i = 0, \ldots, n-1$, with the first and the last vertex being the same, $v_0 = v_n$. A primitive cycle in G consists of the vertices of all edges $e \in E$ that surround one face of $\overline{G}$. The **degree of a face** $f \in F$, $\deg(f)$, is the number of adjacent faces in $\overline{G}$, or equivalently, it is equal to the number of edges that constitute the corresponding primitive cycle in G.

We can now observe the consequences of decimating G into G' on the corresponding dual graphs $\overline{G}$ and $\overline{G'}$.

Theorem 4 *Let $\overline{G}(F, \overline{E})$ be the dual graph of $G(V, E)$, $G'(V', E')$ be a non-overlapping decimated graph of $G(V, E)$, and $\overline{G'}(F', \overline{E'})$ the corresponding dual graph (Fig. 6). For every face $f' \in F'$ there exists a face $f \in F$ such that* $\boxed{\deg(f') \leq \deg(f)}$.

Proof:
Let face $f' \in F'$ have degree $\deg(f') = n$. Then f' corresponds to a primitive cycle $C' = (p_0, p_1, \ldots, p_n)$; $p_i \in V', i = 0, \ldots, n$; $p_0 = p_n$, being composed of n edges $(p_i, p_{i+1}) \in E', i = 0, \ldots, n-1$. From $(p_i, p_{i+1}) \in E'$ follows that there must exist vertices $s_i \in RF(p_i)$