

Homotopic transformations of combinatorial maps

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Abstract. In this contribution, we propose the notion of homotopy for both combinatorial maps and weighted combinatorial maps. We also describe transformations that are homotopic in the defined sense. The usefulness of the concept introduced is illustrated on two applications. The first one consists in calculating a skeleton using homotopic transformations of weighted combinatorial maps. The result is a compact combinatorial map describing the structure of the skeleton without further processing. The second application consists in run length encoding of all the regions described by a combinatorial map. Although these demonstrations are defined on combinatorial maps defined on a square grid, the majors insights of the paper are independant of the embedding.

keywords : Homotopy, skeletonization, combinatorial map

1 Introduction

Homotopy characterizes, in continuous topology, elastic transformations that preserve certain topological properties, transforming a simple arc into a simple arc for example. The definition of homotopy for digital sets has been proposed [13] in order to characterize transformations of such sets preserving topological properties such as the region inclusion tree, or more generally, equivalence classes of paths. The definition of homotopy of transformations on gray-level images has also been proposed [13, 10], as well as on ordered sets [1]. Homotopy is an important concept, as it characterizes topological properties of skeletons, graytone skeletons and watersheds [13, 10, 11, ?].

Combinatorial maps have been introduced as a code for planar graphs. They have been already used in image analysis to encode topological maps with different embeddings [6, 5, 2, 9]. Some transformations of combinatorial maps have been proposed [3].

In this paper, we propose to extend the notion of homotopy to combinatorial maps (section 2), and to weighted combinatorial maps, i.e. combinatorial maps to which a single real number is associated with each dart (section 3), coherently with the classical definition on digital sets. The main advantage is in the

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design of classes of transformations that have nice topological properties, independently of the embedding of the sets studied. Thus, combinatorial maps with different embeddings can be treated with the same classes of transformations and algorithms such that properties defined independently from their embedding are preserved. Some transformations that are homotopic in the defined sense are also presented. Two applications are presented (section 4), putting in evidence the advantages of the proposed definitions and transformations. Defining homotopy on a combinatorial map naturally leads to the definition of a new class of skeletonization algorithms (section 4.1). The second application presented in section 4.2, consists of constructing a combinatorial map encoding conveniently horizontal runs.

2 Combinatorial Maps

2.1 Basic definitions

Let us recall some definitions. A *combinatorial map* is a triplet $G = (\mathcal{D}, \sigma, \alpha)$ where $\mathcal{D}$ is a set of elements called darts (or half-edge), σ and α are two permutations defined on $\mathcal{D}$ such that α is an involution without fixed point :

$$\forall d \in \mathcal{D}, \alpha^2(d) = d$$

An example of a combinatorial map is drawn in Fig. 1. Each dart may be viewed as a directed half-edge of an embedded planar graph, and is associated to a vertex. The darts d and $\alpha(d)$ are associated to a unique edge of the drawn planar graph. σ defines the arrangement of darts turning counterclockwise around a vertex. A combinatorial map can be seen as a graph with explicit orientation around the vertices.

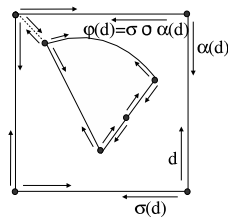


Fig. 1. A combinatorial map

A combinatorial map may be used to encode a topological map, i.e. the partition of an orientable surface into a set of vertices, a set of arcs, and a set of faces. The orbits $\sigma^*(d)$ describe the darts associated with a unique vertex of the represented topological map. The orbits $\alpha^*(d)$ define edges of the map. The orbits $\varphi^*(d)$ of the permutation $\varphi = \sigma \circ \alpha$ define the faces of the encoded topological map.

Each combinatorial map has a well defined dual, which may be viewed as the same topological map where faces and vertices exchange their roles.

Some topological notions such as loops, bridges, etc, can be defined straightforwardly for combinatorial maps [3]. Let us consider a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and one of its darts $d \in \mathcal{D}$.

- $\alpha^*(d)$ is a *self loop* iff $\alpha(d) \in \sigma^*(d)$.
- $\alpha^*(d)$ is a (*dart*) *self-direct loop* iff $\sigma(d) = \alpha(d)$ or $\sigma(\alpha(d)) = d$.
- $\alpha^*(d)$ is a *bridge* iff $\alpha(d) \in \varphi^*(d)$.
- $\alpha^*(d)$ is a *pendant edge* iff $\sigma(d) = d$ or $\sigma(\alpha(d)) = \alpha(d)$.

Paths and cycles can also be defined for combinatorial maps. A *path* of a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ is an ordered sequence of darts $P = (d_1, \dots, d_n)$ such that $\forall i \in \{1, \dots, n-1\}, d_{i+1} \in \sigma^*(\alpha(d_i))$. The *reverse path* $\alpha(P)$ of P is defined by $\alpha(P) = (\alpha(d_n), \dots, \alpha(d_1))$. A *cycle* of a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ is a $P = (d_1, \dots, d_n)$ such that $d_1 \in \sigma^*(\alpha(d_n))$.

We now introduce the concept of connected components defined on a combinatorial map, which will be used later in the text. Connected components are intuitively defined as connected components of the equivalent graph. Consider a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$. The relation R on $\mathcal{D}$ such that :

$d R d'$ means : $\exists P = (d_1, \dots, d_n)$ a path of G with $d \in \alpha^*(d_1)$ and $d' \in \alpha^*(d_n)$.

It can be easily shown that R is an equivalence relation. The *connected components* of G are the equivalence classes of the relation R .

On a plane partition, faces may contain holes, which may be coded by connected components of the corresponding combinatorial map. However, by doing so, we loose the information of inclusion, and the dual graph as introduced above is no longer valid. In order to validate the description, a set of conjugate darts have to be added to the combinatorial map, connecting holes with the external boundary of the faces they are lying in (Fig. 1). These edges form bridges (and no new face is created). We consider in the following text that the holes have been connected with fictive edges. In an application requiring the differentiation of real and fictive edges, darts may be marked.

2.2 Contraction and removal

The *removal* of an edge $\alpha^*(d)$ removes d and $\alpha(d)$ from the initial combinatorial map.

Definition 1. Consider a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and a dart $d \in \mathcal{D}$. The removal of the edge $\alpha^*(d)$ creates the sub-map $G \setminus \alpha^*(d) = (\mathcal{D} \setminus \alpha^*(d), \sigma', \alpha)$ defined by :

$$\left[\begin{array}{l} \sigma'(d') = \sigma(d') \\ \sigma'(\sigma^{-1}(d)) = \sigma(d) \\ \sigma'(\sigma^{-1}(\alpha(d))) = \sigma(\alpha(d)) \end{array} \right] \begin{array}{l} \text{if } d' \in \mathcal{D} \setminus \{\sigma^{-1}(d), \sigma^{-1}(\alpha(d))\} \\ \text{if } \sigma(d) \neq \alpha(d) \text{ and } \sigma(\alpha(d)) \neq d \\ \text{if } \sigma(d) = \alpha(d) \end{array}$$

The *contraction* of an edge $\alpha^*(d)$ transforms a combinatorial map G into a combinatorial map G' where d and $\alpha(d)$ have been removed from the dual $\overline{G'}$.

Definition 2. Consider a combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$ and a dart $d \in \mathcal{D}$. The contraction of the edge $\alpha^*(d)$ creates the sub-map G' defined by :

$$G' = G/\alpha^*(d) = \overline{\overline{G} \setminus \alpha^*(d)}$$

The two transformations are illustrated on the Fig. 2.

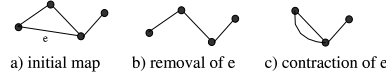


Fig. 2. Removal and contraction transformations

2.3 Homotopy

We introduce first the new notion of homotopy on combinatorial maps, derived from the definitions of [13] p.89.

Definition 3. Consider the set $\mathcal{G}$ of all combinatorial maps. A mapping Φ from $\mathcal{G}$ onto itself is said to be homotopic if it transforms a combinatorial map G into a combinatorial map $\Phi(G)$ such that there is a bijection between the faces and the connected components of G and of $\Phi(G)$.

Theorem 1. The contraction of an edge $\alpha^*(d)$ which is not a self-loop is a homotopic transformation.

Proof. The proof is straightforward, as the contraction does not either remove a face nor creates a new connected component.

Theorem 2. The removal of a pendant edge is a homotopic transformation.

Proof. The transformation preserves also the number of faces and of connected components.

The contraction of an edge which is not a self-loop and the removal of a pendant edge both preserve homotopy.

2.4 Connectivity preserving transformation

In [8], a dual graph is transformed such that the degree of the surviving vertices are preserved. This rule can be applied to a combinatorial map.

Definition 4. A dart d of a combinatorial map is redundant iff $\sigma^2(d) = d$.

Theorem 3. *The contraction of a redundant dart d preserves the cardinality of the orbits of surviving darts.*

Proof. We just give an intuitive idea of the result: Fig. 3 demonstrates the contraction of a redundant dart; as $|\sigma^*(d)| = 2$, the contraction of d does not change $|\sigma^*(\sigma(\alpha(d)))|$ and $|\sigma^*(\alpha(d))|$.

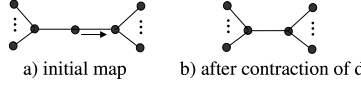


Fig. 3. Contraction of a redundant dart d

Contraction of redundant edges is connectivity preserving. It is a homotopic transformation, as it preserves the number of connected components and faces of a combinatorial map.

3 Weighted combinatorial maps

3.1 Definitions

We introduce here new notions related to combinatorial maps whose darts are associated with a single real number.

Definition 5. *A weighted combinatorial map is a 4-tuple $(\mathcal{D}, \sigma, \alpha, w)$ where :*

- $(\mathcal{D}, \sigma, \alpha)$ defines a combinatorial map,
- $w : \mathcal{D} \rightarrow \mathbb{R}$ is a function defined on $\mathcal{D}$, associating a real number $w(d)$ to each dart $d \in \mathcal{D}$.

Weights associated to darts can take any value, depending on the application. In particular, we can define a weighted combinatorial map with only positive weights. The *removal* and the *contraction* transformations are defined as removal and contraction of combinatorial map, and do not modify the weights of the remaining darts. The following definition restricts the study to a particular class of weighted combinatorial maps, where two opposite darts d and $\alpha(d)$ have opposite weights.

Definition 6. *We say that a weighted combinatorial map $M = (\mathcal{D}, \sigma, \alpha, w)$ is antisymmetric iff $\forall d \in \mathcal{D}, w(d) = -w(\alpha(d))$.*

The following notions interpret weights of darts of a weighted combinatorial map as difference of elevations of connected vertices.

Definition 7. *An upstream path is a path $P = (d_1, \dots, d_n)$ with only positive weights ($\forall i \leq n, w(d_i) > 0$). An downstream path is a path $P = (d_1, \dots, d_n)$ with only negative weights ($\forall i \leq n, w(d_i) < 0$). A plateau path is a path $P = (d_1, \dots, d_n)$ with only null weights ($\forall i \leq n, w(d_i) = 0$).*

Since the weights on opposite darts of a weighted combinatorial map can have any value, the reverse of an upstream path is not necessarily a downstream path. However, for an antisymmetric weighted map, this is the case, as stated by the following theorem :

Theorem 4. *If a weighted combinatorial map $G = (\mathcal{D}, \sigma, \alpha, w)$ is antisymmetric, then the opposite of every upstream path is a downstream path.*

Proof. By definition, the opposite of a path $P = (d_1, \dots, d_n)$ is the path $P' = (\alpha(d_n), \dots, \alpha(d_1))$. Suppose that P is upstream, then $\forall i \leq n, w(d_i) > 0$. As the map is antisymmetric, we have $\forall i \leq n, -w(\alpha(d_i)) > 0$, and P' is downstream.

3.2 Homotopy revisited

We define here the concept of homotopy for weighted combinatorial maps, in such a way that it is coherent with the definition of Serra ([13] p. 187).

Definition 8. *Consider the set $\mathcal{G}$ of all weighted combinatorial maps. A mapping Φ from $\mathcal{G}$ onto itself is said to be homotopic if it transforms a combinatorial map G into a combinatorial map $\Phi(G)$ such that :*

1. *it preserves the number of faces and of connected components,*
2. *it preserves upstream and downstream paths.*

Two weighted combinatorial maps G_1 and G_2 are homotopic iff there exists a homotopic transformation Φ such that $G_2 = \Phi(G_1)$.

In the preceding definition, the path preserving condition (condition 2) has to be understood as : any upstream (downstream) path is transformed into an upstream (downstream) path, possibly empty, and no new upstream (downstream) path is created.

Definition 9. *The contraction of a dart d of an antisymmetric weighted combinatorial map $G = (\mathcal{D}, \sigma, \alpha, w)$ is path-preserving iff $\sigma^*(d) = \{d\}$ (d is a pendant dart) or d is not a self loop and $w(d) \neq 0$ and $\forall d' \in \sigma^*(d), w(d)w(d') \leq 0$ (the weights of d and of any of the darts adjacent to the same vertex have opposite signs).*

The contraction of path preserving darts is a homotopic transformation, as stated in the following theorem.

Theorem 5. *A path-preserving contraction of an antisymmetric weighted combinatorial map G is a homotopic transformation.*

Proof. Condition 1. of definition 8 is already proved in theorem 1. Condition 2. of definition 8 holds for the following arguments : If $\sigma^*(d) = \{d\}$, then d is a pendant dart, and its contraction does not create new upstream or downstream path. Consider an upstream path $P = (d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_n)$. By definition, $\forall j \leq n, w(d_j) > 0$. By contraction of d_i , this path will be transformed into the path

$P' = (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$ which is upstream. Suppose that by contraction of a dart d_i with $w(d_i) \neq 0$, the non-upstream path $P = (d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_n)$ is transformed into the upstream path $P' = (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n)$. Then, as P' is upstream and not P , $\forall j \neq i, w(d_j) > 0$ and $w(d_i) < 0$. As the combinatorial map is antisymmetric, we have $w(\alpha(d_{i-1})) = -w(d_{i-1}) < 0$. Then $w(\alpha(d_{i-1}))w(d_i) > 0$ and the contraction of d_i is not path-preserving. This reasoning can also be used for proving that no downstream path is created.

On Fig. 4, the contraction of a dart which is not path preserving is illustrated. We can remark that a downstream path which did not exist is created (in bold). On Fig. 5, the contraction of the path preserving dart d does not create any

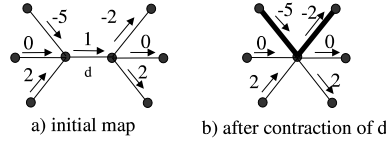


Fig. 4. Contraction of a dart d which is not path preserving

upstream nor downstream path.

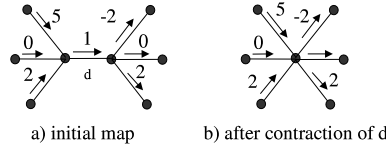


Fig. 5. Contraction of a path preserving dart d

4 Applications

4.1 Gray-tone skeletons

In this section, we apply the preceding treatments to the computation of skeletons of gray-level images, i.e. thin subsets of the crest network of a gray level image [10]. Dual graph contractions invariant to monotonic transformations have been studied in [7]. The framework presented here is different in that it is based on an alternative original graph, and uses the notion of homotopic transformation for combinatorial map discussed above.

For skeletonisation applications, we start by constructing a weighted combinatorial map on a pixel based scale. Then, by applying some transformations

which reduce the number of darts while preserving the homotopy of the combinatorial map until stability is reached, we obtain a compact representation of the skeleton of the original image.

The initial combinatorial map can be obtained from a straightforward algorithm first introduced by M. Pierrot Deseilligny and al. [11]. An image I is defined as a function of digital support $[0, x_{max}] \times [0, y_{max}]$ to $\mathbb{Z}$. We define I' as the image :

$$I'(x, y) = ((I(x, y) \ d_{max} + D(x, y)) \ x_{max} + x) \ y_{max} + y \quad (1)$$

where $D(x, y)$ is the distance of a point (x, y) to the nearest point with lower intensity and $d_{max} = \max_{(x, y)} D(x, y)$ (D can easily be related to the classical distance transform of cross-sections of the original gray level images). I' assigns to each pixel a unique value. The weighted combinatorial map is built using a simple algorithm based on a local analysis of the 8-neighborhood of each pixel p of I' . Each 3x3 neighborhood is decomposed into sets of 4-connected components, the values of which are greater than the value of the central pixel. We construct the contour map by adding an edge (pair of conjugate darts) that connects the central pixel with the highest valued pixel of each component. The weights being associated to the darts are given by the difference between the end-vertex and the origin-vertex of each dart. The map is obviously antisymmetric. One can demonstrate that the combinatorial map is connected, and that a bijection between the faces of the combinatorial map and the local minima of I' exists [11].

Homotopic transformations can then be applied in order to simplify the combinatorial map, and to get rid of the undesired edges. For example, darts that are either redundant or pendant, and which contraction is path-preserving, can be contracted until stability in order to obtain the simplest combinatorial map describing the crest network of the image. Some darts describing relevant features can be excluded from the contraction operation, in order to preserve those features. This is equivalent to defining anchor points [11, 12]. As a classical example of “anchor darts”, we may want to keep pendant darts with negative weights, corresponding to peaks in the image.

Fig. 6 shows the results of path-preserving contractions until stability of the redundant darts (on the right), with pendant darts with negative weight as anchor darts (characterizing peaks in the original image).

The result is a compact combinatorial map describing the structure of the skeleton in a much more compact way than a raster graph. Such a structure, constructed after thinning in [9], is constructed in the first step of the skeletonization algorithm, and may be useful for further processing steps. Moreover, other criteria than homotopy such as geometry, etc, may be considered quite easily. With the used criteria, the result is independant of the order of the contractions. The above scheme also works for different types of grid or digital topologies, by giving a proper embedding of the underlying combinatorial maps.

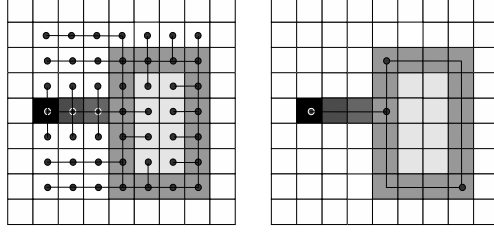


Fig. 6. Skeleton of a weighted combinatorial map

4.2 Curve-based runlength encoding

Rungraphs [4] have been proposed to encode line drawing images. In our application, we use a slightly different representation for encoding horizontal runs describing the interior of regions coded by a combinatorial map in a compact way.

We consider an antisymmetric weighted combinatorial map $G = (\mathcal{D}, \sigma, \alpha, w)$ such that each vertex $\sigma^*(d)$ is a point of $\mathbb{Z}^2$, each edge $\alpha^*(d)$ relates a pair of (4 or 8)-neighbors, and for a dart $d \in \mathcal{D}$, $w(d) = y' - y$ with y' and y the vertical coordinate of respectively the vertex $\sigma^*(\alpha(d))$ and $\sigma^*(d)$.

Horizontal-runs can encode the interiors of the regions of the combinatorial map in a computational convenient way. For some vertical bands, we can describe coherent runs a single set of digital curves, using the curves to deduce the horizontal runs line by line (the bands A and B of Fig. 7.b). The edges of the map correspond to digital curves such that they decrease monotonically. With such a description, an algorithm finding the digital curves needed to draw a region for each row of an image is easily implemented.

Such curves can be found straightforwardly from the previous combinatorial map by contracting its path-preserving redundant darts. Fig. 7 illustrates the result of such an algorithm. On the left, the initial map is drawn. On the right, the contracted map is represented. The regions A and B indicate the rows of the image for which the same set of curves is used to derive the horizontal runs. As

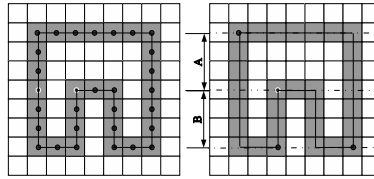


Fig. 7. Curve-based encoding of horizontal runs

in the previous application, other embeddings may be considered.

5 Conclusion and perspectives

In this paper, we have proposed the concept of homotopy for combinatorial maps and weighted combinatorial maps. The main advantage doing so is to define homotopic transformations independently of the embedding. Other advantages appear when considering the computation of skeletons, for example, for which a compact structure is produced. Another application, the efficient encoding of horizontal runs of a partition, has also been presented, which may be viewed in the skeletonization framework.

Other applications are possible. For example, the producing of hierarchies of skeletons is straightforward. We could also consider the extension the proposed framework into higher dimensions.

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