Digital Topologies Revisited: An Approach Based on the Topological Point-Neighbourhood*

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Abstract. Adopting the point-neighbourhood definition of topology, which we think may in some cases help acquire a very good insight of digital topologies, we unify the proof technique of the results on 4-connectedness and on 8-connectedness in $\mathbb{Z}^2$. We also show that there is no topology compatible with 6-connectedness. We shortly comment on potential further use of this approach.

Keywords: Image processing, digital topology, adjacency, path-connectedness and topological connectedness in $\mathbb{Z}^2$.

1 Introduction and Basic Definitions

The domain of a *digital picture* can be viewed as a subset of $\mathbb{Z}^2$, where $\mathbb{Z}$ stands for the set of integers, together with some adjacency neighbourhood structure [2, 7, 6] assigned to each point. Thus, for instance, we may talk on 8-adjacency neighbourhood structure (in shorthand, 8-structure) if each point $(x, y) \in \mathbb{Z}^2$ is given the adjacency neighbourhood

$$\begin{matrix} (x-1, y-1) & (x, y-1) & (x+1, y+1) \\ (x, y-1) & (x, y) & (x, y+1) \\ (x-1, y+1) & (x+1, y) & (x+1, y+1) \end{matrix}$$

In the sequel, let us assume as plausible that *the adjacency neighbourhood structure is homogeneous* (i.e., for each (x_0, y_0), the natural translation $(x, y) \to (x + x_0, y + y_0)$ is an adjacency isomorphism) and *symmetric* with respect to the point (x, y).

* This work was supported by a grant from the Austrian National Fonds zur Förderung der wissenschaftlichen Forschung (No. S7002MAT).

We would like to demonstrate that the point-neighbourhood definition of topology adopted here provides us with a good method for deciding when (if) there is a topology compatible with adjacency. We will present different proofs of previously known results (see [2, 6, 7, 9]) and slightly extend them.

We also assume that the adjacency neighbourhood never exceeds the 8-neighbourhood, but note the result holds also for any larger neighbourhood.

In the following we give some elementary definitions. In *section* 2 we present the different *adjacency neighbourhood structures*. *Section* 3 contains the main results. Finally some conclusions are given in *section* 4.

1.1 Path-Connectedness in $\mathcal{S}$

Suppose that $\mathbb{Z}^2$ is given a (homogeneous and symmetric) neighbourhood structure. In order to refer to this structure, let us call it $\mathcal{S}$. Suppose that p, q are points of $\mathbb{Z}^2$. By an $\mathcal{S}$-*path from* p *to* q we understand a finite sequence of points $p = p_1, p_2, \ldots, p_n = q$ such that p_i is a neighbour of p_{i-1} $(1 < i \leq n)$ in the structure $\mathcal{S}$. Let us call the points p, q $\mathcal{S}$-*related* if there is an $\mathcal{S}$-path from p to q.

Let X be a subset of $\mathbb{Z}^2$. Since the relation of being $\mathcal{S}$-related, for a given $\mathcal{S}$, is obviously an equivalence relation on X. This relation gives rise to a partition of X into classes of $\mathcal{S}$-related elements. Let us call each class of this equivalence an $\mathcal{S}$-*component*.

A natural question arises ([8, 2], etc.) if (when) the $\mathcal{S}$-components can be obtained as the components of a connectedness relation of a topology. Let us view basic notions we need for pursuing this question. Out of several possible definitions of (classical) topology, the definition involving the point-neighbourhood structure may best serve the purpose.

1.2 Topological Connectedness

Definition (Topological space): *Let P be a set. Let us assign to each $x \in P$ a set $\mathcal{U}(x)$ of subsets of P which is subject to the following conditions:*

(i) *if $U \in \mathcal{U}(x)$, then $x \in U$,*

(ii) *if $U \in \mathcal{U}(x)$ and $U \subset V$, then $V \in \mathcal{U}(x)$,*

(iii) *if $U, V \in \mathcal{U}(x)$, then $U \cap V \in \mathcal{U}(x)$,*

(iv) *if $U \in \mathcal{U}(x)$, then there exists $V \in \mathcal{U}(x)$ such that, for each $y \in V$, $U \in \mathcal{U}(y)$.*

The set P together with the assignment $\mathcal{U}(x)$, for each $x \in P$, is called a topological space. *The sets $U \in \mathcal{U}(x)$ are called* topological neighbourhoods *of x. We denote the assignment $x \to \mathcal{U}(x)$ by t - the topology. So we can refer to the couple (P, t), meaning the corresponding* topological space.

It should be noted that this definition of topological space is equivalent to the "base–for–open–sets" definition or to the "closure" definition. This can be easily verified by a straightforward translation (see e. g. [1]). One should also observe that it is the axiom (iv) which is usually responsible for inconveniences when one looks for "suitable" topologies (this can be compensated by a possibility to apply the topological result back into the real model).

Let (P, t) be a topological space defined in the sense of previous definition (i. e. via point–neighbourhoods) and let X be a subset of P. We say that (X, t_1) is a *topological subspace* of (P, t) if, for each $x \in X$, the set $\mathcal{U}(x) \cap X$ is the set of all neighbourhoods of x in the topology t_1. It can be seen easily that (X, t_1) is then a topological space in its own right. Moreover, if (P, t) is a topological space and Y, V are subsets of P with $Y \subset V$, then if $(Y, t_1), (V, t_2)$ are topological subspaces of (P, t), then (Y, t_1) is a topological subspace of (V, t_2).

Definition (Topological connectedness): *Let (P, t) be a topological space. We say that (P, t) is* disconnected *if there are two disjoint sets R, S such that $R \cup S = P$ and, moreover, for each $r \in R$ and each $s \in S$ the set R is a neighbourhood of r and S is a neighbourhood of s. The space (P, t) is said to be* connected *if it is not disconnected. Finally, a subset X of (P, t) is said to be connected if the subspace (X, t_1) of (P, t) is connected.*

Obviously, a connected topological space may have plenty of disconnected subspaces. For instance, the set $(-\infty, 0) \cup (0, +\infty)$ or the set $\{1, 2, 3, \ldots\}$ are obviously disconnected subspaces of the space R of reals (resp. the topology induced by one of the equivalent n-norms $n = 1, 2, \infty$). Also, the set Q of all rational numbers is also disconnected – we can write $Q = Q_1 \cup Q_2$, where $Q_1 = \{q \in Q \mid q < \sqrt{2}\}$ and $Q_2 = \{q \in Q \mid q > \sqrt{2}\}$. The last example shows that the question on deciding about connectedness of a subspace may be sometimes nontrivial.

2 Neighbourhood Structures on $\mathbb{Z}^2$

Definition (Compatibility): *Let $\mathcal{S}$ denote a given adjacency structure on $\mathbb{Z}^2$ and let t be a topology on $\mathbb{Z}^2$. We say that t* is compatible with *$\mathcal{S}$ if for each $X, X \subset \mathbb{Z}^2$, the set X is connected (with respect to t) if and only if X is $\mathcal{S}$-connected.*

Let us employ the following definition (see also [3]). Let (P, t) be a topological space. We say that (P, t) is *locally finite* if each point $x \in P$ possesses the finite neighbourhood $U \,\epsilon\, \mathcal{U}(x)$, i.e. $|U| < \infty$. Obviously, if (P, t) is locally finite then *each point $x \in P$ possesses a smallest neighbourhood* in t and this neighbourhood is finite. This follows from the closedness of the neighbourhood under the formation of intersections. An important fact in our considerations is this: If $U(x)$ is the smallest (topological) neighbourhood of x, then $U(x)$ must also be a neighbourhood of all points in $U(x)$. This immediately follows from the point-neighbourhood definition of topology (the axiom (iv)).

We will now investigate all possible (homogeneous and symmetric) adjacency neighbourhood structures in $\mathbb{Z}^2$ (see the figure below) and formulate results on the compatible topologies. We will in turn take up the 0-adjacency, 2-adjacency, 4-adjacency, 6-adjacency and 8-adjacency.

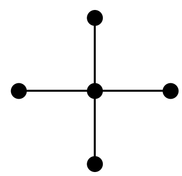
(i)

0-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the only point (x, y) is adjacent)

(ii)

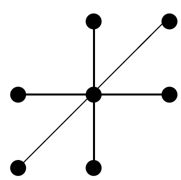
2-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x-1, y)$ and $(x+1, y)$ are adjacent)

(iii)



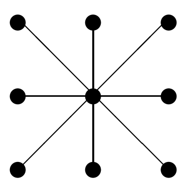
4-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x-1, y), (x+1, y), (x, y-1), (x, y+1)$ are adjacent)

(iv)



6-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x-1, y), (x+1, y), (x, y-1), (x, y+1)$, $(x-1, y-1), (x+1, y+1)$ are adjacent)

(v)



8-adjacency (to each point $(x, y) \in \mathbb{Z}^2$ the points $(x-1, y), (x+1, y), (x, y-1), (x, y+1)$, $(x-1, y-1), (x+1, y+1)$, $(x-1, y+1)$, $(x+1, y-1)$ are adjacent).

Remark: The n-adjacencies, n is odd, are not homogeneous and 2-point connections (1-adjacency) can be compared only with a trivial topology like the discrete topology.

3 Compatible Topologies on $\mathbb{Z}^2$

Theorem 1: *There is exactly one topology which is compatible with the 0-adjacency. This topology is the discrete topology on* $\mathbb{Z}^2$.

Proof: The discrete topology making each point $(x, y) \in \mathbb{Z}^2$ a neighbourhood of (x, y) is clearly compatible with 0-adjacency.

Theorem 2: *There are infinitely many topologies which are compatible with the 2-adjacency in $\mathbb{Z}^2$. Among those there are infinitely many, which are locally finite.*
Proof: For a fixed $x \in \mathbb{Z}$, we obviously have a topology, t_x, on the subspace $T_x = \{(x, y) \mid y \in \mathbb{Z}\}$ which is compatible with the 2-adjacency on the set T_x. Indeed, it suffices to take $\mathcal{U}(x) = \{U \subset T_x \mid \{(x, y)\} \subset U\}$ provided y is odd, $\mathcal{U}(x) = \{U \subset T_x \mid \{(x, y-1), (x, y), (x, y+1)\} \subset U\}$ provided y is even. Note that the above neighbourhoods contain two types as smallest and finite neighbourhood (*i.e. the 1-dimensional* Marcus-Wyse *topology*).

Since the roles of odd and even numbers are obviously interchangeable and since the adjacency connectedness (2-connectedness) of "different levels" do not affect each other, we can take, for each $x \in \mathbb{Z}$, one of the two topologies on T_x, obtaining a topology that is compatible with 2-adjacency. Since we have infinitely many combinations at our disposal, the result is proved.

Remark (not locally finite topologies): A question of separate purely topological curiosity may arise whether we can construct, for the 2-adjacency, a topology which is compatible with the adjacency and which is not locally finite. This seems to be possible – the standard ultrafilter construction can be applied in this case or more easily the Frechet-filter of the infinite intervals $\{x \epsilon \mathbb{Z} \,|x \geq z\}$, $z \epsilon \mathbb{Z}$. These topologies are, however, hardly relevant to digital pictures.

Let us now consider the 4-adjacency. The following result, which is due to [9] and [6] is in force. We will show how one obtains this result in the point–neighbourhood formalism. The little auxiliary results stated as Observations 3.1, 3.2 may be of certain value in their own right.

Theorem 3: *There are 2 topologies which are compatible with the 4-adjacency — the 2-dimensional* Marcus-Wyse *topologies τ [9]:*

$$U \in \tau \quad \equiv \quad \begin{cases} U(x, y) & : \text{ if } x + y \text{ is even} \quad (\text{resp. odd}), \\ \{(x, y)\} & : \text{ else}, \end{cases}$$

with $U(x, y) = \{(x, y), (x, y-1), (x, y+1), (x+1, y), (x-1, y)\}$.

Proof: The result easily follows from the following two observations.

Observation 3.1: Each topology which is compatible with 4-adjacency is locally finite. Moreover, if $U(x, y)$ is the smallest neighbourhood of (x, y) in a topology compatible with 4-adjacency, then

$$U(x, y) \quad \subseteq \quad \{(x, y), (x, y-1), (x, y+1), (x+1, y), (x-1, y)\}.$$

Proof: Let $(x, y) \in \mathbb{Z}^2$ and let us show that the point (x, y) must have a finite neighbourhood.

Let $X_1 = \{(x, y)\}$ and $X_2 = \{(u, v) \in \mathbb{Z}^2 \mid |x - u| + |y - v| \geq 2\}$, see Fig. 1.

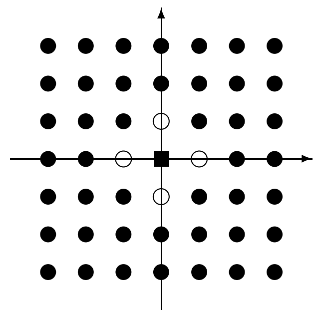


Fig. 1. The two sets $X_1 = \{\blacksquare\}$ and $X_2 = \{\bullet\}$

Then both X_1 and X_2 are 4-connected but $X_1 \cup X_2$ is not. It follows that there is a topological neighbourhood of (x, y) in any topology compatible with 4-adjacency which is disjoint with X_2. In other words, there is a topological neighbourhood, $U(x, y)$, such that $U(x, y) \subseteq \mathbb{Z}^2 - X_2$. This is what we wanted to show.

Observation 3.2: Let $(x, y) \in \mathbb{Z}^2$ and let t be a topology compatible with 4-adjacency. Let $U(x, y)$ be the smallest neighbourhood of (x, y) in t. Then either $U(x, y) = \{(x, y)\}$ or $U(x, y) = \{(x, y), (x, y - 1), (x, y + 1), (x + 1, y), (x - 1, y)\}$.
Proof: By the previous observation, $U(x, y) \subseteq \{(x, y), (x, y - 1), (x, y + 1), (x + 1, y), (x - 1, y)\}$. Suppose that $U(x, y) \neq \{(x, y)\}$. Then, without a loss of generality, $(x - 1, y) \in U(x, y)$. Suppose now that $(x, y + 1) \notin U(x, y)$ (again, one argues analogously if there is another edge than $(x, y+1)$ outside of $U(x, y)$). Since the set $\{(x-1, y), (x, y+1)\}$ is not 4-connected and the set $\{(x, y), (x, y+1)\}$ is 4-connected, we infer that the smallest neighbourhood of $(x, y + 1)$, some set $U(x, y + 1)$, must not contain the point $(x - 1, y)$ and must contain the point (x, y). Consequently considering other edges analogously,

$$U(x, y) \cap U(x, y + 1) = \{(x, y)\}.$$

But since $U(x, y+1)$ is a topological neighbourhood, it must be also a topological neighbourhood of the point (x, y). It follows that $U(x, y) \cap U(x, y + 1)$ must be a topological neighbourhood of the point (x, y).
But $U(x, y) \cap U(x, y + 1) = \{(x, y)\}$ which is a contradiction. This completes the proof of Observation 3.2.

Let us return to the proof of Theorem 3. Let t be a topology which is compatible with the 4-adjacency. It is obvious that the singleton sets $\{(x, y)\}$ cannot constitute the neighbourhoods for all $(x, y) \in \mathbb{Z}^2$ (we would obtain the discrete topology; the discrete topology is obviously not compatible with 4-adjacency). It is also obvious that the sets

$$\{(x, y), (x-1, y), (x+1, y), (x, y-1), (x, y+1)\}$$

cannot be the smallest neighbourhoods for all $(x, y) \in \mathbb{Z}^2$ (we would not have a topology at all). Thus, in every topology compatible with 4-adjacency we must have, for some points, both the singleton smallest neighbourhoods and the "star–like" neighbourhoods. Having found the necessary conditions for a compatible topology, the rest consists of an easy inductive argument already presented in [9].

Choose, for instance, the point $(0, 0)$. Then either the set $U_1(0, 0) = \{(0, 0), (-1, 0), (1, 0), (0, 1), (0, -1)\}$ or the set $U_2(0, 0) = \{(0, 0)\}$ must be the smallest neighbourhood of $(0, 0)$.
In the former case, the smallest neighbourhoods of the points $(-1, 0), (1, 0), (0, -1), (0, 1)$ must necessarily be the singleton sets, the smallest neighbourhoods of the points $(-1, -1), (-1, 1), (1, -1), (1, 1)$ must be the "star" sets, and so on. In the latter case, the smallest neighbourhoods of the points $(-1, 0)$, $(1, 0)$, $(0, -1)$, $(0, 1)$ must be the "star" sets, the smallest neighbourhoods of the points $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$ must be the singleton sets, and so on.
Consequently, there are only two topologies on $\mathbb{Z}^2$ which are compatible with the 4-adjacency – either the Marcus-Wyse topology or the topology obtained from it by the shift $(x, y) \to (x, y+1)$. The Marcus-Wyse topology allows for a simple description as also the present consideration shows: the smallest neighbourhood of (x, y) is a singleton set provided $x+y$ is even, and the smallest neighbourhood of (x, y) is a star set provided $x + y$ is odd.
The rest would consist in checking that, indeed, the Marcus-Wyse topology is compatible with 4-adjacency. This is not difficult and has been done in detail in [9]. The proof is complete.

Let us now consider the 6-adjacency (see the schema in the figure (iv)). In this case the search for compatible topology would be in vain.

Theorem 4: *There is no topology on $\mathbb{Z}^2$ which is compatible with the 6-adjacency.*
Proof: Suppose that t is a topology compatible with the 6-adjacency. By the very same reasoning we employed in Observations 3.1, 3.2 we can derive the following results:

(i) The topology t is locally finite,

(ii) If $(x, y) \in \mathbb{Z}^2$, then the smallest neighbourhood of (x, y) in t is either the singleton set $\{(x, y)\}$ or the whole 6-star set

$$\{(x, y), (x-1, y), (x+1, y), (x, y-1), (x, y+1), (x-1, y-1), (x+1, y+1)\}.$$

There must be a point in $\mathbb{Z}^2$ with the proper 6-adjacency neighbourhood. Let us denote it again by (x, y). It follows that the points $(x+1, y)$ and $(x+1, y+1)$ must have the singleton set neighbourhoods. This is absurd since the set $\{(x+1, y), (x+1, y+1)\}$ is 6-connected. The proof is complete.

The following corollary to the previous result can be viewed as another proof of the result by Chassery [2] and L. Latecki [7].

Theorem 5: *There is no topology on $\mathbb{Z}^2$ which is compatible with the 8-adjacency.*

Proof: It is easily seen that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two adjacency relations on $\mathbb{Z}^2$ and $\mathcal{S}_2$ is finer then $\mathcal{S}_1$, then the absence of a locally finite compatible topology for $\mathcal{S}_1$ implies the absence of a locally finite compatible topology for $\mathcal{S}_2$. Since there is no topology compatible with 6-adjacency, there is no topology compatible with the 8-adjacency.

4 Conclusion

We have completed the tour over all possible "nice" adjacencies in $\mathbb{Z}^2$. Presumably the next step is testing the suitability of the point–neighbourhood approach is the examination of concrete (finite) configurations of points in $\mathbb{Z}^2$ and, of course, the digital topologies in $\mathbb{Z}^n$ for $n \geq 3$. We intend to pursue this elsewhere. It should be observed, however, that it does not seem possible to analyze $\mathbb{Z}^n$ with the help of viewing $\mathbb{Z}^n$ as a Cartesian product of $\mathbb{Z}^m$ for m smaller than n. This can be graphically seen even for $\mathbb{Z}^2$. Indeed, none of the adjacencies on $\mathbb{Z}^2$ with the exception of the discrete one is obtained as a "Cartesian product" of adjacencies on $\mathbb{Z}$. We may however obtain nontrivial (non-homogeneous) adjacencies this way or, in other words, homogeneity may be a too restricting property for digital topologies. In particular if the data stem from a projection of a higher dimensional space (e.g. 3-dim) onto a lower dimension (e.g. 2-dim). In the case of digital images, we may seek the topological properties of the 3-dim objects in the 2-dim image rather than establishing adjacency across object boundaries. Such occluding boundaries represent discontinuities of the image function and adjacent pixels may correspond to 3-dim points of different objects which are far apart in reality.
If, for instance, we take the 2-adjacency on $\mathbb{Z}$ and multiply it with each other, we obtain an adjacency on $\mathbb{Z}^2$ which is the 4-adjacency on the points of the type $(even, even)$, the 2-adjacency on the points of the type $(even, odd)$ and $(odd, even)$ – in the former case vertically and in the latter case horizontally, and the 1-adjacency on the points of the type (odd, odd). Since the 2-adjacency on $\mathbb{Z}$ allows for a compatible topology, so does our "mixed" adjacency on $\mathbb{Z}^2$, which directly leads to the abstract cellular complexes of Kovalevsky [3, 4].

This survey is part of an ongoing research project with two primary goals to extend the results to *higher dimensions* and to *irregular adjacency neighbourhood structures* [5].

References

1. E. Cech: *Topological Spaces*, Interscience, Wiley, New York, 1966.
2. J. M. Chassery: *Connectivity and consecutivity in digital pictures*, Computer Graphics and Image Processing 9, 294–300, 1979.
3. E. Khalimsky, R. Kopperman and P. R. Meyer: *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. 36, 117, 1980.
4. V. A. Kovalevsky: *Finite topology as applied to image analysis*, Computer Vision, Graphics and Image Processing 46, 141-161, 1989.
5. W. G. Kropatsch: *Equivalent Contraction Kernels and the Domain of Dual Irregular Pyramids*, Technical Report PRIP-TR-042, TU-Vienna, 1996.
6. L. Latecki: *Digitale und Allgemeine Topologie in der bildhaften Wissensrepräsentation*, Ph.D.-Thesis, Hamburg, 1992.
7. L. Latecki: *Topological connectedness and 8-connectedness in digital pictures*, Computer Vision, Graphics and Image Processing: Image Understanding 57, 261–262, 1993.
8. A. Rosenfeld: *Digital topology*, Am. Math. Monthly 86, 621–630, 1979.
9. F. Wyse and D. Marcus et al.: *Solution to Problem 5712*, Am. Math. Monthly 77, 1119, 1970.