Nearness in digital images, and proximity spaces *

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Abstract

The concept of "nearness", which has been dealt with as soon as one started studying digital images, finds one of its rigorous forms in the notion of proximity space. It is this notion, together with 'nearness preserving mappings", that we investigate in this paper. We first review basic examples as they naturally occur in digital topologies, making also brief comparison studies with other concepts in digital geometry. After this we characterize proximally continuous mappings in metric spaces. Finally, we show by example that the 'proximite complexity" of a finite covering in a ditigal picture may be too high to be adequately depicted in a finite topological space. This combinatorial result indicates another conceptual advantage of proximities over topologies.

1 Introduction

Recently there have been quite an intense investigation of topological structures in image processing, mostly in connection with the analysis of connectivity and the operation of thinning (see e.g. [2, 4, 9, 13], etc.). An interesting attempt to introduce richer structures than those of topology, and replacing thus 'local" continuity properties by a global notion of nearness, has been done in [12] where the authors contemplated the so called semi–proximity spaces as a theoretical tool in the image processing studies. In this paper we want to go on in a similar vain by shedding light on certain questions which implicitely announced themselves in the paper [12], and by complementing the results of [12] with some new findings.(Our investigation here is relatively technical. The reader is supposed

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to be acquainted with the motivation for investigating digital topology and thus for the selection of the problems we pursue. Reading the papers [8, 12, 18] would certainly be instrumental as it was for the authors. We hope, however, that our exposition is reasonably selfcontained.)

The paper is organized as follows: In section 2 proximity spaces are introduced and related to topological spaces, in particular to the discrete topology introduced by Marcus and Wyse. Section 3 investigates the proximities in metrical spaces. In section 4 we discuss the problem whether nearness of finite partitions implies a finite topology on the index set. The conclusion summarizes the paper.

2 Proximity spaces (in relation to topologies and metrics)

We shall deal with proximity spaces as implicitely defined by Riesz [16] and intensely pursued by several authors ([3, 5, 6, 14]). Our definition is a modified version of the definition by E. Čech [3].

Definition 1 (Proximity Space) *A pair (X, π) is called a proximity space if X is a set and π is a binary relation on the power set of X, $\exp X$, which is subject to the following requirements ($A, B, C, D, E \in \exp X$, the symbol A nonπ B means that $A \,\pi\, B$ is false):*

(i) $(A \cup B) \,\pi\, C \Leftrightarrow A \,\pi\, C$ *or* $B \,\pi\, C$,

(ii) $A \,\pi\, B \Rightarrow A \neq \emptyset$ *and* $B \neq \emptyset$,

(iii) $A \cap B \neq \emptyset \Rightarrow A \,\pi\, B$,

(iv) $A \,\pi\, B \Rightarrow B \,\pi\, A$,

(v) if $\{a\} non\pi B$, then there is a set A such that $a \in A$ and, moreover, $\{a\} non\pi\ (X - A)$ and $A non\pi\ B$.

The axioms of a proximity space reflect the properties we observe when we consider the common–sense nearness. It should be noted that the only less plausible axiom of proximity – the axiom *(v)* – guarantees that a proximity induce a topology (the semi-proximities of [12] induce only closure spaces). There is an important link of proximity spaces and topological spaces. Here is the precise formulation of this fact (the proof can be found in [3] but it is routine and can be easily done.

Theorem 1 (Proximity Space, Topological Space)

(i) Let (X, π) be a proximity space. Let $\bar{A} = \{x \in X |\ \{x\} \,\pi\, A\}$. Then $(X, ^-)$ is a topological space.

(ii) Let $(X, ^-)$ be a topological space, where $^-$ means the topological closure on $\exp X$*. Let $A \,\pi\, B \Leftrightarrow \bar{A} \cap \bar{B} \neq \emptyset$. Then (X, π) is a proximity space.*

Thus, by the previous result, we can associate topologies to proximities. However, many proximities on X may induce the same topology on X (and, vice versa, a topology may give rise to several proximities which induce it). For the reader's intuition, let us note that e. g. the following two proximities p_1, p_2 on an infinite set X define the same (discrete) topology: $A\, p_1\, B \Leftrightarrow A \cap B \neq \emptyset$, $A\, p_2\, B \Leftrightarrow$ either $A \cap B \neq \emptyset$ or both the sets A and B are infinite.

It is possible to adopt the notion of proximity as primary and view the notion of topology as secondary. One of the reasons for doing it is that the topology describes only *the local character of points* in a picture (or, if we want, the proximity of points and sets). But if we are to treat geometrical qualities of a picture – a situation which typically arises in image processing – it is *the proximity of sets* which matters most. E.g. in the case of scanned text documents the characters are the connected components of the black pixels of the thresholded image. The pixel sets of adjacent characters of words are closer to each other than between the words.

Thus, besides the concrete reason obvious from the study of thinning and shape deformation [12], the general reason for investigating the proximity relation lies in its fundamental role in all kinds of geometrically oriented considerations.

In view of Theorem 1 and the investigation of connectedness in digital images [20], the following proximity in Z^2 is worth recording. Recall that if $[r, s] \in Z^2$, then by the 4–neighbourhood of $[r, s]$ we mean the set $\{[r, s], [r \pm 1, s], [r, s \pm 1]\}$.

Definition 2 (Marcus–Wyse proximity) *Let Z^2 denote the subset of R^2 consisting of all points with the integer coordinates. Let $A \subset Z^2, B \subset Z^2$. Let us write $A \,\pi\, B$ if either $A \cap B \neq \emptyset$ or there exists a point $p = [r, s] \in A \cup B$ such that the following assertion holds true:*

(i) $r + s$ is an odd number,

(ii) if $p \in A$, then the 4–neighbourhood of p intersects B,

(iii) if $p \in B$, then the 4–neighbourhood of p intersects A.

The result of [20] (see also [11]) can be now reformulated proximity-wise. Following [12], let us agree to say that a set S in a proximity space (P, π) is connected if S cannot be written in the way $S = A \cup B$, where A, B are two nonproximal sets (i.e. A *non*π B).

Theorem 2 *The graph–theoretic connectedness in Z^2 induced by the 4–neighbourhood adjacency relation coincides with the proximite connectedness induced by the Marcus-Wyse proximity.*

Let us consider a metric space, (M, ρ).Then (M, ρ) can be viewed as a topological space with the closure $\bar{A} = \{x \in M \mid \rho(x, A) = 0\}$. In this way the metric space (M, ρ)

induces a "topological" proximity, π_t, defined as follows: $A\ \pi_t\ B \Leftrightarrow \bar{A} \cap \bar{B} \neq \emptyset$. But (M, ρ) also induces a metric proximity, π_m. Write, for two subsets A and B, $\rho(A, B) = = \inf_{a \in A, b \in B} \rho(a, b)$ and set $A\ \pi_m\ B \Leftrightarrow \rho(A, B) = 0$. It is easily seen that if $A\ \pi_t\ B$, then $A\ \pi_m\ B$ but not necessarily the other way round. If, for instance, A is the graph in R^2 of the function $f(x) = \frac{1}{x}$ and B is the x–axis in R^2, then $A\ \pi_m\ B$ with respect to the Euclidean metric, but not $A\ \pi_t\ B$.

For a reader not especially trained in topology, let us explicitely clarify why (and when) the latter phenomen (of the difference of π_t and π_m) may occur. Recall that a metric space M is called *compact* ([3, 5], etc.) if each sequence in M allows for a convergent subsequence in M. As known, a subset P of R^n endowed with the Euclidean metric (taken form R^n) is compact if and only if it is closed and bounded.

Theorem 3 *Let (M, ρ) be a compact metric space. Then $\pi_t = \pi_m$, i.e. in compact metric spaces the topological proximity agrees with the metric proximity.*

Proof : Suppose that $A, B \subset M$. It only remains to show that if $A\ \pi_m\ B$, then $A\ \pi_t\ B$, the other implication is always valid. Let $A\ \pi_m\ B$. We are to show that $\bar{A} \cap \bar{B} \neq \emptyset$. The relation $A\ \pi_m\ B$ means $\rho(A, B) = 0$. It follows that for each $n \in N$ we can find points $a_n \in A$, $b_n \in B$ such that $\rho(a_n, b_n) \leq \frac{1}{n}$. By compactness, there is a subsequence, a_{n_k}, of $\{a_n\}$ which converges to some element $a \in M$, and there is a subsequence, b_n, of $\{b_{n_k}\}$ which converges to some $b \in M$. Obviously, $a = b$ and therefore the sequences $\{a_{n_\ell}\}$ and $\{b_{n_\ell}\}$ converge to a common element $a (= b)$. Since $a \in \bar{A}$ and $b \in \bar{B}$ and since $a = b$, we see that $\bar{A} \cap \bar{B} \neq \emptyset$. This completes the proof. ∎

The morphisms in the category of proximity spaces are the proximally continuous mappings.

Definition 3 (proximally continuous mapping) *Let (X_1, π_1), (X_2, π_2) be proximity spaces. A mapping $f : X_1 \to X_2$ is called proximally continuous if the following implication is true:*

If $A, B \subset X_1$ and $A\ \pi_1\ B$, then $f(A)\ \pi_2\ f(B)$.

Thus, proximally continuous mappings are those mappings which preserve proximity. It is easily seen that a proximally continuous mapping is automatically continuous when understood as a mapping between the respective topological spaces induced by proximities. A continuous mapping does not have to be proximally continuous even if we consider it with respect to the metric proximity.

Example: Consider the function $f(x) = x^2 : (0, +\infty) \to (0, +\infty)$. This mapping is obviously continuous but not proximally continuous. Indeed, let p_n be such a sequence that $(p_n + \frac{1}{2})^2 - p_n^2 \geq 1$. Let $A = \{p_n \mid n \in N\}$ and $B = \{p_n + \frac{1}{n} \mid n \in N\}$. Then $A\ \pi_m\ B$ but $f(A) non\, \pi_m\ f(B)$.

3 Metric proximities

In this paragraph we are going to prove that the metric proximity of a space determines, up to a metric equivalence, the metric of the space. This relatively deep result has found applications in a number of geometric problems (see e. g. [5] for relevant comments). This result can be expressed in terms of 'small" (=countable) subsets of the metric space in question and therefore it may have bearing on digital geometry. Also, we see that metric considerations of digital pictures ([17, 18, 19]) have a proximity character (i.e. can be expressed in proximity terms).

Let us take up the proof of the result. We provide transparent proof based on elementary reasonings only. We also point out other features of metric spaces relevant to proximity. Recall first two standard definitions.

Definition 4 *Let (M, ρ) be a metric space and let $\{a_n\}$ be a sequence in M. We say that $\{a_n\}$ is a* **Cauchy sequence** *if for each $\varepsilon > 0$ there is $n_0 \in N$ such that $\rho(x_m, x_n) \leq \varepsilon$ provided $n \geq n_0, m \geq n_0$. We say that a sequence$(x_n)_{n \in N}$ in M is* **metrically discrete** *(of order ε) if for any $n, m \in N$ we have $\rho(x_m, x_n) \geq \varepsilon$.*

The following proposition is essential in our argument. It may be interesting in its own right. Before we formulate it, let us agree to call the set $B_r(a) = \{b \in M \mid \rho(a, b) \leq r\}$ the r-ball around a.

Proposition 1 (Sequence principle in metric spaces) *Each sequence in a metric space contains either a Cauchy sequence or a metrically discrete subsequence.*

Proof : Take a sequence in a metric space and form the collection of all 1 -balls centered at each of its points. If each of these balls contains only finitely many points of the sequence, we can easily construct a subsequence of the given sequence which is metrically discrete of order 1. It not, there is a 1-ball around a point of the sequence which contains infinitely many points of the sequence. Take these points and form the collection of all $\frac{1}{2}$-balls centered at these points. If each of these $\frac{1}{2}$-balls contains only finitely many points, we can easily construct a metrically discrete subsequence of order $\frac{1}{2}$. If not, there is a point such that the $\frac{1}{2}$-ball around it contains infinitely many points. Going on this way inductively, either the procedure stops at the n-th step and we have an $\frac{1}{n}$-discrete subsequence, or we obviously obtain a Cauchy subsequence of the given sequence. ■

We shall need one more metric notion. Let us recall it together with proximal continuity in metric space.

Definition 5 *Let (M_1, ρ_1), (M_2, ρ_2) be metric spaces. Let $f \colon M_1 \to M_2$ be a mapping. In accord with our general definition, we say that f is* **proximally continuous** *if the following property is fulfilled:*

If P and Q are subsets of M_1 such that $\rho_1(P, Q) = 0$, then $\rho_2(f(P), f(Q)) = 0$.

We say that f is **metrically continuous** *if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\rho_1(x, y) < \delta$, then $\rho_2(f(x), f(y)) < \varepsilon$.*

Let us now formulate and prove the main result of this paragraph. In the effort to make the proof accessible for nonspecialists in topology, we use only elementary reasonings. The novelty seems to be the utilization of the sequence principle as established in Prop. 1.

Theorem 4 *A mapping between metric spaces is proximally continuous if and only if it is metrically continuous.*

A consequence: If two metrics on a set induce the same proximity, they have to be (metrically) equivalent.

Proof : A metrically continuous mapping between metric spaces is obviously proximally continuous. Let us take up the nontrivial implication of the theorem.

Let $f\colon M_1 \to M_2$ be proximally continuous. Suppose f is not metrically continuous. It means that for some $\varepsilon > 0$ there exists sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ in M_1 so that $\rho_1(a_n, b_n) \to 0$ whereas $\rho_2(f(a_n), f(b_n)) \geq \varepsilon$. Let us look at the sequence $(f(a_n))_{n\in N}$. If it contains a Cauchy subsequence, then there is an infinite subset of this sequence which is all contained in an $\frac{\varepsilon}{2}$-ball. Let us denote this subsequence by $(a_{n_k})_{k\in N}$. Then the sets $(a_{n_k}\colon k \in N)$ and $(b_{n_k}\colon k \in N)$ are obviously $\frac{\varepsilon}{2}$-apart (the triangular inequality), which is absurd. If there is a Cauchy subsequence of $(f(b_n))_{n\in N}$, we can apply an analogous reasoning. Suppose that neither of the former two cases implies. By the sequence principle, we can easily construct metrically discrete subsequences $(f(a_{n_k}))_{k\in N}$ and $(f(b_{n_k}))_{k\in N}$ of $f(a_n)_{n\in N}$ and $f(b_n)_{n\in N}$, respectively. For simplicity, let us denote them again by $f(a_n)_{n\in N}$ and $f(b_n)_{n\in N}$.

Let us assume that these sequences are metrically discrete of order α. Define $r = \frac{1}{2}\min\{\varepsilon, \alpha\}$ and form the collection of all the r-balls centered at $f(a_n)_{n\in N}$. Note that each of them contains at most one of the elements $f(b_n)_{n\in N}$. Let us now proceed inductively. Put $n_1 = 1$. Take $n_2 \in N$, $n_2 > 1$ such that

$n_2 > h$ if $f(b_h)$ belongs to the r-ball centered at $f(a_1)$, and

$n_2 > k$ if the r-ball centered at $f(a_k)$ contains $f(b_1)$,

(if neither h or k exists, we simply take $n_2 = n_1 + 1$). By induction, given $n_1 < n_2 < \cdots < n_i$, take $n_{i+1} \in N$, $n_{i+1} > n_i$ such that

$n_{i+1} > h$ if $f(b_h)$ belongs to the r-ball centered at $f(a_{n_i})$, and

$n_{i+1} > k$ if the r-ball centered at $f(a_k)$ contains $f(b_{n_i})$.

Note that, by our construction, if $j \in \{n_k | k \in N\}$, then $f(b_j)$ does not belong to any of the r-balls centered in $f(a_{n_k})$. The sets $f(a_{n_k})_{k\in N}$ and $f(b_{n_k})_{k\in N}$ are therefore apart of the order r. Write $A = \{a_{n_k}\}_{k\in N}$, $B = \{b_{n_k}\}_{k\in N}$. The $\rho_1(A, B) = 0$ but $\rho_1(f(A), f(B)) \geq r$. This means that f is not proximally continuous - a contradiction. This completes the proof. ■

The main consequence of the latter theorem (and its proof) as far as the potential application in image processing goes brings the next theorem. It does not seem to be explicitely formulated in the literature but right this formulation may have relevant bearing on the digital images studies – it reduces proximities of general sets to proximities of small (=countable, digitally accessible) sets. (Recall that a set is said to be countable if it has the smallest possible infinite cardinality, i.e. if it has the cardinality of natural numbers.)

Theorem 5 *Let ρ_1, ρ_2 be two metrics on a set M. If $\rho_1(A,B) = 0 \Leftrightarrow \rho_2(A,B) = 0$ for all* **countable** *subsets of M, then the metrics ρ_1, ρ_2 are equivalent. In other words, if the proximities given by ρ_1 and ρ_2 agree when restricted to countable subsets of M, then ρ_1 and ρ_2 are metrically indistinguishable.*

4 Near and far sets in a finite partition – could 'nearness" be controlled by a finite topology?

Each picture can be viewed as a partition of the underlying set into a finite family of sets. The main point in understanding the partition is specifying which sets are 'near" (proximal) and which are 'far" (non-proximal). In this paragraph we exhibit an example which shows that the proximal relation of sets in a finite partition may be too complex to be described with the help of notion of finite topology. This relates in a natural way our investigation here with topological studies in image processing (see [13, 8, 11, 18, 10, 20], etc.). (Recall that by a finite partition of a set S we mean a mutually disjoint collection $\{S_k\ (k = 1, 2, \ldots, n)\}$ of non-empty subsets of S such that $\bigcup_{k=1}^{n} S_k = S$.)

Let us now introduce an auxiliary notion.

Definition 6 *Let (X, π) be the metric proximity space of the metric space (M, ρ). Let $P = \{S_k \mid k = 1, 2, \ldots, n\}$ be a partition of X. Let us say that P is* **controlled by topology** *if there is a topology t on the set $T = \{1, 2, \ldots, n\}$ such that $S_k\ \pi\ S_\ell$ if and only if the set $\{k, \ell\}$ is topologically connected as a topological subspace of T.*

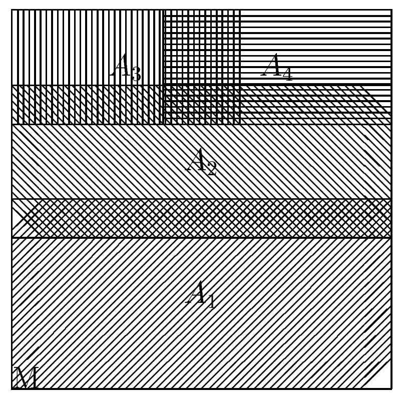
If each partitioncould be topologically controlled, which seems conceivable at first sight, we would find ourselves in an advantages situation in view of the understanding of finite topologies (see e.g. [18] and [10]). However, it is not necessarily the case. We will illustrate it by an example. Before, observe that we can restrict ourselves to open sets of the partition if we allow for a small degree of overlapping (in practice, we have to allow for it anyhow in view of the imperfection of our measurement).

Proposition 2 *Let (M, ρ) be a metric space. Let $\{S_k \mid k = 1, 2, \ldots, n\}$ be a partition of M. Then there is an ε, $\varepsilon > 0$ such that the following statement holds true: If $T_k = \{x \in M \mid \rho(x, S_k) < \varepsilon\}, k = 1, 2, \ldots, n$, then each T_k is an open subset of M, and $T_k \cap T_\ell \neq \emptyset$ if and only if $S_k\ \pi_\rho\ S_\ell$.*

Proof : All $T_k,\ k = 1, \ldots, n$ are obviously open. Further, consider all couples S_k, S_ℓ such that $S_k\ non\,\pi_\rho\ S_\ell$. Since there are only finitely many couples with this property, there is a sufficiently small ε, $\varepsilon > 0$ such that $T_k \cap T_\ell = \emptyset$ for all couples of T_k, T_ℓ corresponding to the couples S_k, S_ℓ considered. Since we have $S_k \ \pi_m \ S_\ell$ for the remaining couples, we infer from the definition of metric proximity that the corresponding neighbourhood sets T_k, T_ℓ fulfil $T_k \cap T_\ell \neq \emptyset$. The proof is complete. ■

We see from the above proposition, that we can simply construct the counterexample on the disjointness – intersection basis of an open covering.

Example: Let $M \subset R^2$, $M = \{(x, y) \mid 0 < x < 10, 0 < y < 10\}$. Let $A_1 = \{(x, y) \in M \mid x \in M, y < 5\}$, $A_2 = \{(x, y) \in M \mid 4 < y < 8\}$, $A_3 = \{(x, y) \in M \mid 0 < x < 6, 7 < y\}$, $A_4 = \{(x, y) \in M \mid 4 < x, 7 < y\}$. Then the proximity of the sets A_1, A_2, A_3, A_4 cannot be topologically controlled. Indeed, considering the couples of the sets which are disjoint (resp. which overlap), we see that the topological space on $\{1, 2, 3, 4\}$ which would control the covering would have to have precisely the following subsets connected: $\{1, 2\}, \{2, 3\}, \{2, 4\}$ and $\{3, 4\}$. But the main result of [13] says that this is impossible.



5 Conclusions

We suggest that the notion of proximity of sets might be a useful tool for theoretical studies in image processing. We have made some initial steps towards justifying this opinion. We have exhibited basic examples and we linked them with previous topological investigations (for instance, with the Marcus–Wyse topologies). Then we analysed more thoroughly the metric proximities. As a main result we showed that the metric proximity of small sets determines in a way the metric of the underlying space. The interpretation of this result reads, roughly, that if we can verify which sequences of points in a metric space are proximal and which are not, we can in a certain sense reconstruct the metric. In the end we showed that a topological result known from the investigation of 8-adjacency relation in Z^2 disproves a conjecture about 'topologizing" proximities in a partition.

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