Reduction Factors of Pyramids on Undirected and Directed Graphs [*]

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Abstract

We present two new methods to determine contraction kernels for the construction of graph pyramids. The first method is restricted to undirected graphs and yields a reduction factor of at least 2.0. This means that with our method the number of vertices in the subgraph induced by any set of contractible edges is reduced to half or less by a single parallel contraction. Our second method also works for directed graphs. In case of stochastic pyramids, the second method yields even higher reduction factors than the first one in all our tests.

1 Introduction

In a regular image pyramid (for an overview see [9]) the number of pixels at any level l, is r times higher than the number of pixels at the next reduced level $l + 1$. The reduction factor r is greater than one and it is the same for all levels l. If s denotes the number of pixels in an image I, the number of new levels on top of I amounts to $log_r(s)$. Thus, the regular image pyramid may be an efficient structure to access image objects in a top-down process.

However, regular image pyramids are confined to globally defined sampling grids and lack shift invariance [1]. In [10] it was shown how these drawbacks can be avoided by irregular (stochastic) image pyramids. Each level represents a partition of the pixel set into cells, i.e. subsets of 4-connected pixels. The construction of an irregular image pyramid is iteratively local [10] [6]:

- The cells have no information about their global position.

- The cells are connected only to (direct) neighbors.
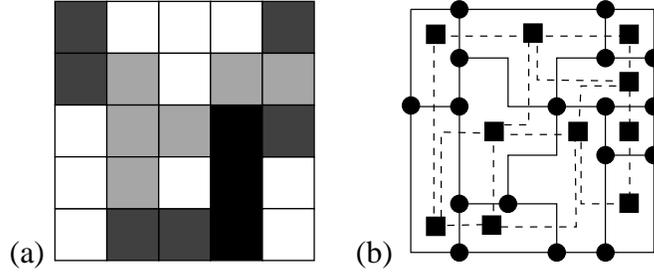
Figure 1: (a) Partition of pixel set into cells. (b) Representation of the cells and their neighborhood relations by a dual pair $(\overline{G}, G)$ of plane graphs. $\overline{G}$ has square vertices and dashed edges. G has circular vertices and solid edges.

- The cells cannot distinguish the spatial positions of the neighbors.

On the base level (level 0) of an irregular image pyramid the cells represent single pixels and the neighborhood of the cells is defined by the 4-connectivity of the pixels. A cell on level $l + 1$ is a union of neighboring cells on level l. Two cells c_1 and c_2 are neighbors if there exist pixels p_1 in c_1 and p_2 in c_2 such that p_1 and p_2 are 4-neighbors (Figure 1a,b). We assume that any two successive levels are different, i.e. that at least two neighboring cells in the lower level have been united. In particular, there exists a highest level h . Furthermore, we restrict ourselves to irregular pyramids with an apex, i.e. level h contains only one cell.

In this paper we will represent the levels as dual pairs $(\overline{G_l}, G_l)$ of plane graphs $\overline{G_l}$ and G_l. The vertices of $\overline{G_l}$ represent the cells on level l and the edges of $\overline{G_l}$ represent the neighborhood relations of the cells on level l (Figure 1b). The edges of G_l represent the borders of the cells on level l, possibly including so called pseudo edges needed to represent neighborhood relations to cells enclosed by other cells. Finally, the vertices of G_l represent meeting points of at least three edges from G_l. The sequence $(\overline{G_l}, G_l)$, $0 \le l \le h$ is called graph pyramid. The plan of the paper is as follows. In Section 2 we will give the main idea of the stochastic pyramid algorithm and in Section 2.1 we will see that graph pyramids from maximal independent vertex sets may have a very poor reduction factor (arbitrarily close to 1.0). Moreover, experiments show that poor reduction factors are likely, especially when the images are large. We propose two modifications. The one in Section 3 guarantees a reduction factor of 2.0, but is applicable only if the edges may be contracted in both directions. The modification proposed in Section 4 also works in case of constraints on the directions. This modification yields the highest reduction factors in the case of stochastic graph pyramids, in all our tests.

2   Maximal Independent Vertex Set

In the following the iterated local construction of the (stochastic) irregular image pyramid in [10] is described in the language of graph pyramids. The main idea is to first calculate a so called *maximal independent vertex set* [3]. Let the vertex set and edge set of of $\overline{G_l}$ be denoted by $\overline{V_l}$ and $\overline{E_l}$, respectively. The incidence relation of $\overline{V_l}$, denoted by $\overline{\iota_l}(\cdot)$ maps each edge
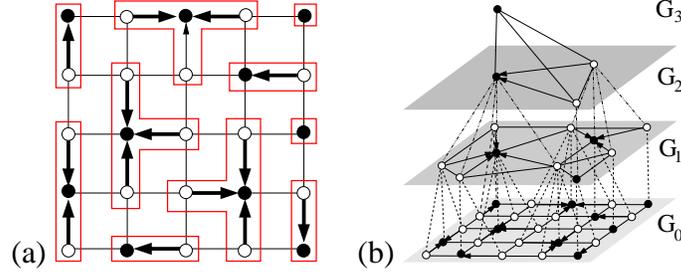
Figure 2: (a) The black vertices form a maximal independent vertex set. The frames indicate a corresponding collection of contraction kernels. (b) A graph pyramid from maximal independent vertex sets.

from $\overline{E_l}$ to its set of end vertices. The neighborhood $\Gamma_l(\overline{v})$ of a vertex $\overline{v} \in \overline{V_l}$ is defined by

$$\Gamma_l(\overline{v}) = \{\overline{v}\} \cup \{\overline{w} \in \overline{V_l} \mid \exists \overline{e} \in \overline{E_l} \text{ such that } \overline{v}, \overline{w} \in \overline{\iota_l}(\overline{e})\}.$$

A subset $\overline{W_l}$ of $\overline{V_l}$ is called maximal independent vertex set if:

1. $\overline{w_1} \notin \Gamma_l(\overline{w_2})$ for all $\overline{w_1}, \overline{w_2} \in \overline{W_l}$,

2. for all $\overline{v} \in \overline{V_l}$ there exists $\overline{w} \in \overline{W_l}$ such that $\overline{v} \in \Gamma_l(\overline{w})$.

An example of a maximal independent vertex set is shown in Figure 2a. Maximal independent vertex set (MIS) [10] [11] may be generated as follows.

MIS Algorithm:

1. Mark every element of $\overline{V_l}$ as *candidate*.

2. Iterate the following two steps as long as there are candidates.

   (a) Assign random numbers to the candidates of $\overline{V_l}$.

   (b) Determine the candidates whose random numbers are greater than the random numbers of all neighboring candidates and mark them as *member* (of the maximal independent set) and as *non-candidate*. Also mark every neighbor of every new member as *non-candidate*.

3. In each neighborhood of a vertex that is not a member there will now be a member. Let each non-member choose its neighboring member, say the one with the maximal random number (we assume that no two random numbers are equal).

The assignment of the non-members to their members determine a collection of *contraction kernels*: each non-member is contracted towards its member and all contractions can be done in a single parallel step. In Figure 2a the contractions are indicated by arrows. A graph pyramid from maximal independent vertex sets can be seen in Figure 2b. Note that we remove parallel edges and self-loops that emerge from the contractions, if they are not needed to encode inclusion of regions by other regions (in the example of Figure 2b we do not need loops nor parallel edges). This can be done by dual graph contraction [7].
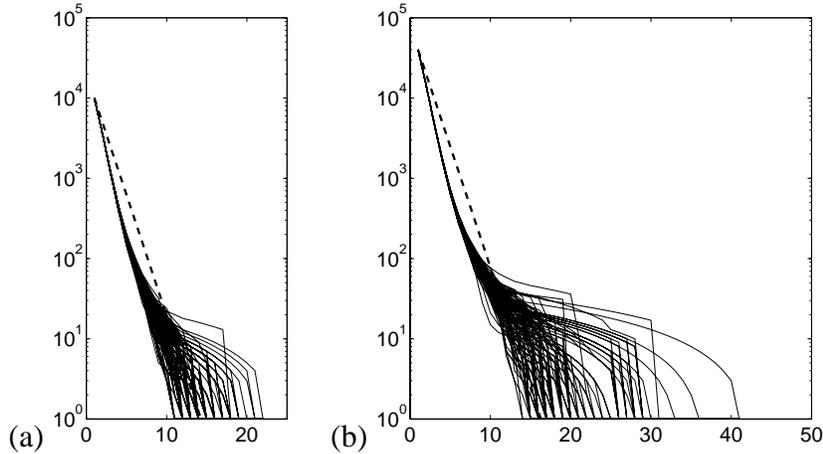
Figure 3: MIS Algorithm: Number of vertices (y-axis) for the graph of size (a) 100×100, and (b) 200×200. x-axis: levels of the graph pyramid. The slope of the lines depicts the reduction factor. Solid lines for test results and dashed line for reduction factor 2.0.

2.1 Experiments with Maximal Independent Vertex Sets

Uniformly distributed random (u.d) values are assigned to the vertices in the base level graphs. We generated 1000 graphs, on top of which we built stochastic graph pyramids. In our experiments, Section 2.1, Section 3.1 and Section 4.1, we used graphs of size 10000 and 40000 vertices, which correspond to image sizes of 100×100 and 200×200 pixels, respectively. Solid lines in Figure 3, 6 and 9 depict the first 100 of 1000 tests. Data in Table 1 were derived using graphs of size 200×200 vertices with 1000 experiments.

The numbers of levels needed to reduce the graph at the base level (level 0) to a graph consisting of a single vertex (top of the pyramid) are given in Figure 3 (a),(b). From Figure 3 we see that the height of the pyramid cannot be guaranteed to be logarithmic, except for some good cases. In the worst case the pyramid had 22 levels for 100×100 vertices and 41 levels for the graph with 200×200 vertices, respectively. Poor reduction factors are likely, as can be seen in Figure 3, especially when the images are large. This is due to the evolution of larger and larger variations between the vertex degrees in the contracted graphs (Table 1). The absolute maximum in-degree was 148. The *apriori* probability of a vertex being the local maximum is dependent of its neighborhood. The larger the neighborhood the smaller is the *apriori* probability that a vertex will survive. The number of iterations necessary to complete the maximum independent set per level (iterations for correction [10]) are the same as reported by [10].

To summarize, a constant reduction factor higher then 1.0 cannot be guaranteed and bad cases have a high probability, as can be seen in Figure 3.
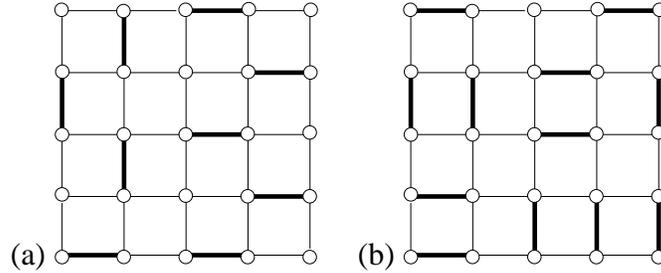
Figure 4: (a) A maximal matching. (b) A matching with more edges than in (a).

3 How to guarantee a Reduction Factor of 2.0

In the following we aim at a collection $\mathcal{C}$ of contraction kernels in a plane graph $\overline{G}$ such that

- each vertex of $\overline{G}$ is contained in exactly one kernel of $\mathcal{C}$, and

- each kernel $\mathcal{C}$ contains at least two vertices.

We assume that $\overline{G}$ is connected. Clearly, the contraction of all kernels in $\mathcal{C}$ will reduce the number of vertices to half or less. In contrast to [10] we start with independent **edge** sets or *matchings*, i.e. edge sets in which no pair of edges has a common end vertex. The selection of $\mathcal{C}$ is done in three steps.

  MIES Algorithm:

1. A maximal matching M of edges from $\overline{G}$ is determined.

2. M is enlarged to a set M^+ that induces a spanning subgraph of $\overline{G}$.

3. M^+ is reduced to $\mathcal{C}$.

In the first step, a maximal matching may be determined by a iteratively local process as specified in the Section 2. Note that a maximal matching of $\overline{G}$ is equivalent to a maximal independent vertex set on the edge graph of $\overline{G}$ [4]. Since M is only required to be maximal, the edge set M cannot be enlarged by another edge from $\overline{G}$ without loosing independence. As can be seen in Figure 4(a), a maximal matching M is not necessarily maximum: there may be a matching M' that contains more edges than M.

The collection of contraction kernels defined by a maximal matching M may include kernels with a single vertex. Let v denote such an isolated vertex (isolated from M) and choose a non-self-loop e that has v as an end vertex. Since M is maximal, the end vertex $w \neq v$ of e belongs to an edge that is contained in the matching. Let M^+ denote the set of edges that are in M or that are chosen to connect isolated vertices to M (the second step of MIES). The subgraph of $\overline{G}$ that is induced by M^+ spans $\overline{G}$ and its connected components are trees of depth one or two (Figure 5(a)). A tree of depth two can be separated into two trees of depth one each by removing the unique edge, both end vertices if which belong to other edges of the tree (Figure 5(b)) (the third step of MIES). Still, each vertex of $\overline{G}$ belongs to a tree (of depth one). The arrows in
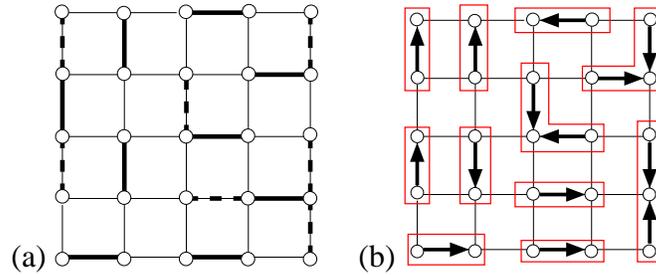
Figure 5: (a) The matching from Figure 4a enlarged by connecting formerly isolated vertices to the maximal matching. (b) After breaking up trees of depth two into trees of depth one. The arrows indicate possible directions of the contractions.

Figure 5b indicate possible directions of contractions. Note that in case of kernels with more than one edge the directions within the kernel cannot be chosen independently of one another. This is why the proposed method cannot be extended to applications in which there are a priori constraints on the directions of the contractions. However, the proposed method works for the stochastic case (no preconditions on edges to be contracted) and for connected component analysis, where the attributes of the end vertices are required to be identical.

3.1 Experiments with Maximal Independent Edge Sets

The numbers of levels needed to reduce the graph at the base level to a graph consisting of a single vertex are shown in Figure 6 (a),(b). The experiments show that the reduction factor, even in the worst case, is always bigger than the theoretical lower bound 2.0, indicated by the
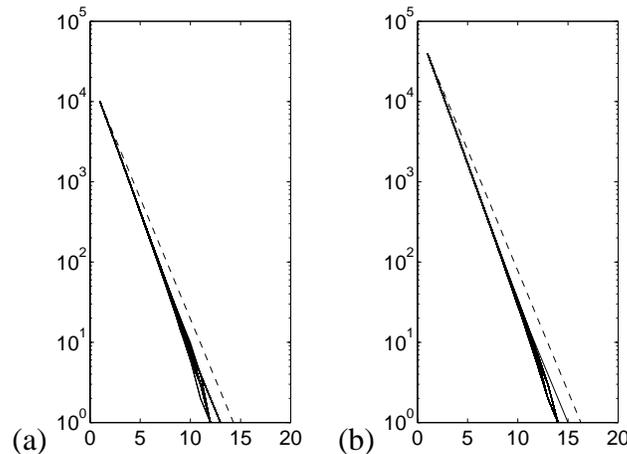
Figure 6: MIES Algorithm: Number of vertices (y-axis) for the graph of size (a) 100×100, and (b) 200×200. x-axis: number of levels. The slope of the lines depicts the reduction factor. Solid lines for test results and dashed line for reduction factor 2.0.

dashed line in Figure 6. This method is more stable than MIS. As can be seen in Figure 6, the variance of the slope is smaller then in case of MIS (Figure 3). The mean number of iteration for correction per level was higher for MIES (Table 1).

4 Constraints on the Directions of the Contractions

In many graph pyramid applications such as line image analysis [2, 8] and the description of image structure [5] a directed edge e with source u and target $v \neq u$ must be contracted (from u to v), only if the attributes of e, u, and v fulfill a certain condition. In particular, the condition depends on u being the source and v being the target. The edges that fulfill the condition are called *preselected* edges. From now on the plane graphs in the pyramid have directed edges. Typically, the edges in the base level of the pyramid form pairs of reverse edges, i.e. for each edge e with source u and target v there exists an edge e' with source v and target u. However, the set of preselected edges may contain e without containing e'. The goal is to build contraction kernels with a "high" reduction factor from the set of preselected edges. The reduction will always be determined according to the directed graph induced by the preselected edges. For example, if the number of vertices in the induced subgraph is reduced to half, the reduction factor will be 2.0. From the example in Figure 7a it is clear that, in general, no reduction factor larger than 1.0 can be guaranteed. We require that the contraction kernels are vertex disjoint rooted trees of depth one or zero (single vertices), each edge of which is directed towards the root. A set $\overline{C}$ of directed edges forms such a collection of contraction kernels if and only if $\overline{C}$ contains none of the edge pairs depicted in Figure 7b. Seen from a directed edge e with source u and target $v \neq u$ that one wants to contract (from u to v), no edge $e' \neq e$ with end vertex (source or target) equal to u or source equal to v may be contracted. An edge e together with those edges that one may not contract if e is contracted form a neighborhood $N(e)$ of e. Figure 8a depicts $N(e)$ in case of u and v both having 4 neighbors. To find a **maximal** (independent) set of directed edges (MIDES) forming vertex disjoint rooted trees of depth zero or one, we proceed analogously to the generation of maximal independent vertex sets, as explained in the Section 2. Let $\overline{E_l}$ denote the set of directed edges in the graph $\overline{G_l}$ of the graph pyramid. We proceed as follows.
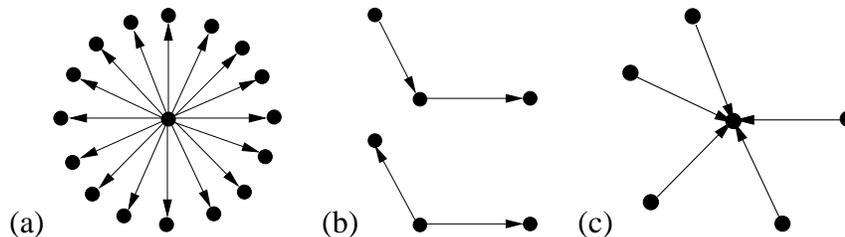
Figure 7: (a) The reduction factor of a star with n edges pointing away from the center is $(n+1)/n$. (b) Forbidden pairs of directed edges. (c) A legal configuration of directed edges
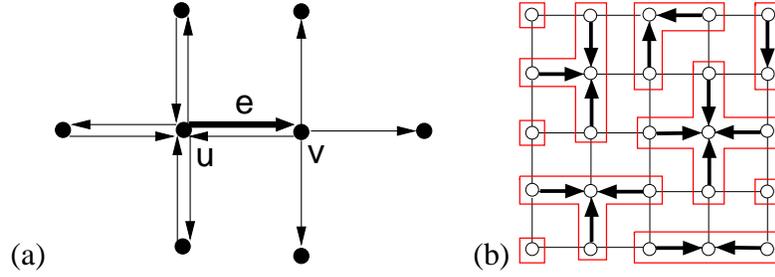
Figure 8: (a) The neighborhood $N(e)$. (b) Maximal independent edge set with respect to $N(e)$.

Process	$\mu(max)$	$\sigma(max)$	$\mu(\#iterations)$	$\sigma(\#iterations)$
MIS	70.69	23.88	2.95	0.81
MIES	11.74	0.71	4.06	1.17
MIDES	13.29	1.06	2.82	1.07

Table 1: Mean μ and standard deviation σ of maximum vertex degrees of the pyramids; Mean μ and standard deviation σ of number of iterations to complete maximum independent set per level of the pyramid.

MIDES Algorithm:

1. Mark every directed edge of $\overline{E_l}$ as *candidate*.

2. Iterate the following two steps as long as there are candidates.

   (a) Assign random numbers to the candidates.

   (b) Determine the candidates $\overline{e}$ whose random numbers are higher (larger) than the random numbers in $N(\overline{e}) \setminus \{\overline{e}\}$ and mark them as *member* (of a contraction kernel). Also mark every $\overline{e}' \in N(\overline{e})$ of every new member $\overline{e}$ as *non-candidate*.

4.1 Experiments with Maximal Independent Directed Edge Sets

Pictures in Figure 9 show the number of levels required to get on top of the pyramid. We see that the reduction factor is better than 2.0 (dashed line) even in the worst case. Also the in-degrees of the vertices is much smaller (13.29) than for MIS (70.69). For the case of the graph with size 200×200 vertices, MIDES needed 13 levels in comparison to 15 levels in the worst case of MIES. The number of iterations needed to complete the maximum independent set was comparable with the one of MIS (Table 1). The MIDES algorithm shows a better reduction factor than MIES, as can be seen in Figure 9.
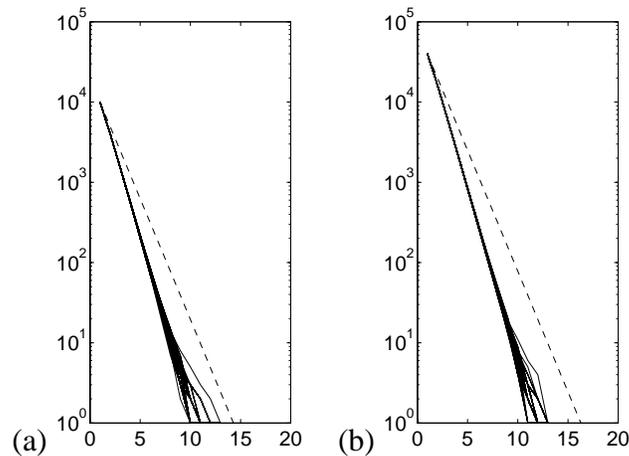
Figure 9: MIDES Algorithm: Number of vertices (y-axis) for the graph of size (a) 100×100, and (b) 200×200. x-axis: number of levels. The slope of the lines depicts the reduction factor. Solid lines for test results and dashed line for reduction factor 2.0

5   Conclusion

Experiments with stochastic decimation using maximal independent vertex sets (MIS) showed a problematic behavior on large images. After an initial phase of strong reduction, the reduction decreases dramatically. This is due to the evolution of larger and larger variations between the vertex degrees in the contracted graphs. To overcome this problem we proposed a method, MIES, based on matchings which guarantees a reduction factor of 2.0. As in the case of independent vertex sets, the method based on matchings does not allow to control the directions of the contractions. The second method, MIDES, that we proposed and tested is based on directed edges and allows to control the directions of the contractions. The experiments showed a non-decreasing reduction that was even stronger than the one obtained from the method based on matchings. Future work will focus on understanding and proving the good performance of the method based on directed edges.

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