Logarithmic Tapering Graph Pyramid *

Yll Haxhimusa[1] and Roland Glantz[2] and Maamar Saib[1]
and Georg Langs[1] and Walter G. Kropatsch[1]

[1] Pattern Recognition and Image Processing Group 183/2
Institute for Computer Aided Automation
Vienna University of Technology
Favoritenstrasse 9, A-1040 Vienna, Austria
[2] Dipartimento di Informatica
Università di Ca' Foscari di Venezia
Via Torino 155, 30172 Mestre (VE), Italy

Abstract. We present a new method to determine contraction kernels for the construction of graph pyramids. The new method works with undirected graphs and yields a reduction factor of at least 2.0. This means that with our method the number of vertices in the subgraph induced by any set of contractible edges is reduced to half or less by a single parallel contraction. Our method yields better reduction factors than the stochastic decimation algorithm, in all tests. The lower bound of the reduction factor becomes crucial with large images.

1 Introduction

In a regular image pyramid (for an overview see [KLB99]) the number of pixels at any level l, is r times higher than the number of pixels at the next reduced level $l+1$. The so called reduction factor r is greater than one and it is the same for all levels l. If s denotes the number of pixels in an image I, the number of new levels on top of I amounts to $log_r(s)$. Thus, the regular image pyramid may be an efficient structure to access image objects in a top-down process.

However, regular image pyramids are confined to globally defined sampling grids and lack shift invariance [BCR90]. In [MMR91,JM92] it was shown how these drawbacks can be avoided by irregular image pyramids, the so called adaptive pyramids. Each level represents a partition of the pixel set into cells, i.e. connected subsets of pixels. The construction of an irregular image pyramid is iteratively local [Mee89,Jol02]:

- the cells have no information about their global position.
- the cells are connected only to (direct) neighbors.
- the cells cannot distinguish the spatial positions of the neighbors.

* This paper has been supported by the Austrian Science Fund under grants P14445-MAT and P14662-INF

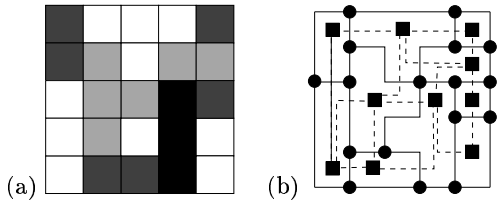

Fig. 1. (a) Partition of pixel set into cells. (b) Representation of the cells and their neighborhood relations by a dual pair $(\overline{G}, G)$ of plane graphs.

On the base level (level 0) of an irregular image pyramid the cells represent single pixels and the neighborhood of the cells is defined by the 4-connectivity of the pixels. A cell on level $l+1$ (parent) is a union of neighboring cells on level l (children). This union is controlled by so called contraction kernels. Every parent computes its values independently of other cells on the same level. This implies that an image pyramid is built in $O[log(image\_diameter)]$ time. For more in depth on the subject see the book of Jolion [JR94]. Neighborhoods on level $l+1$, are derived from neighborhoods on level l. Two cells c_1 and c_2 are neighbors if there exist pixels p_1 in c_1 and p_2 in c_2 such that p_1 and p_2 are 4-neighbors, Figure 1(a). We assume that on each level $l+1$ $(l \geq 0)$ there exists at least one cell not contained in level l. In particular, there exists a highest level h . Furthermore, we restrict ourselves to irregular pyramids with an apex, i.e. level h contains one cell.

In this paper we represent the levels as dual pairs $(\overline{G_l}, G_l)$ of plane graphs $\overline{G_l}$ and G_l, Figure 1(b). The vertices of $\overline{G_l}$ represent the cells on level l and the edges of $\overline{G_l}$ represent the neighborhood relations of the cells on level l, depicted with square vertices and dashed edges in Figure 1(b). The edges of G_l represent the borders of the cells on level l, depicted with solid lines in Figure 1(b), possibly including so called pseudo edges needed to represent the neighborhood relation to a cells completely surrounded by a cell. Finally, the vertices of G_l, the circles in Figure 1(b), represent meeting points of at least three edges from G_l, solid lines in Figure 1(b). The sequence $(\overline{G_l}, G_l)$, $0 \leq l \leq h$ is called (dual) graph pyramid.

The aim of this paper is to combine the advantage of regular pyramids (logarithmic tapering) with the advantages of irregular graph pyramids (their purely local construction and shift invariance). The aim is reached by exchanging the selection method for contraction kernels proposed in [Mee89] by another iteratively local method that now guarantees a reduction factor of 2.0. Experiments with both selection methods show that:

- the old method does not lead to logarithmic tapering graph pyramids, as opposed to our method, i.e. the reduction factors of graph pyramids built by the old method can get arbitrarily close to 1.0.
- the sizes of the receptive fields from the new method are much more uniform.

Not only stochastic decimation [Mee89], but also connected component analysis [KM95] gains from the new method.

The plan of the paper is as follows. In Section 2 we recall the main idea of the stochastic pyramid algorithm and in Section 2.2 we see that graph pyramids from maximal independent vertex sets may have a very small reduction factor. We propose a new method in Section 3, which guarantees a reduction factor of 2.0.

2 Maximal Independent Vertex Set

In the following the iterated local construction of the (stochastic) irregular image pyramid in [Mee89] is described in the language of graph pyramids. The main idea is to first calculate a so called *maximal independent vertex set*[1] [Chr75]. Let $\overline{V_l}$ and $\overline{E_l}$ denote the vertex set and the edge set of $\overline{G_l}$, respectively and let $\iota(\cdot)$ be the mapping from an edge to its set of end vertices. The neighborhood $\Gamma_l(\overline{v})$ of a vertex $\overline{v} \in \overline{V_l}$ is defined by

$$\Gamma_l(\overline{v}) = \{\overline{v}\} \cup \{\overline{w} \in \overline{V_l} \mid \exists \overline{e} \in \overline{E_l} \text{ such that } \overline{v}, \overline{w} \in \iota(\overline{e})\}.$$

A subset $\overline{W_l}$ of $\overline{V_l}$ is called maximal independent vertex set if:

1. $\overline{w_1} \notin \Gamma_l(\overline{w_2})$ for all $\overline{w_1}, \overline{w_2} \in \overline{W_l}$,
2. for all $\overline{v} \in \overline{V_l}$ there exists $\overline{w} \in \overline{W_l}$ such that $\overline{v} \in \Gamma_l(\overline{w})$.

An example of a maximal independent vertex set is shown with black vertices in Figure 2(a), the arrows indicate a corresponding collection of contraction kernels.

2.1 Maximal Independent Vertex Set Algorithm (MIS)

The maximal independent vertex set (MIS) problem was solved using heuristic in [Mee89]. The number of iterations to complete maximal independent set converges in most of the cases very fast, so called iterations for correction [Mee89]. MIS may be generated as follows.

1. Mark every element of $\overline{V_l}$ as *candidate*.
2. Iterate the following two steps as long as there are candidates:
   (a) Assign random numbers to the candidates of $\overline{V_l}$.
   (b) Determine the candidates whose random numbers are larger than the random numbers of all neighboring candidates and mark them as *member* (of the maximal independent set) and as *non-candidate*. Also mark every neighbor of every new member as *non-candidate*.
3. In each neighborhood of a vertex that is not a member there will now be a member. Let each non-member choose its neighboring member, say the one with the maximal random number (we assume that no two random numbers are equal).

[1] also called maximal stable set; we distinct maximal from maximum independent set, which construction is NP-complete.

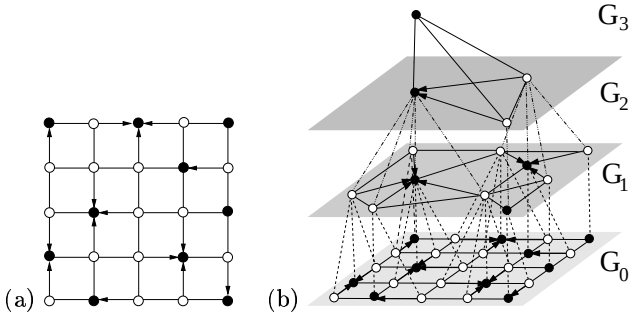

Fig. 2. (a) Maximal independent vertex set. (b) A graph pyramid from maximal independent vertex set.

The assignment of the non-members to their members determines a collection of *contraction kernels*: each non-member is contracted toward its member and all contractions can be done in a single parallel step. In Figure 2(a) the contractions are indicated by arrows. A graph pyramid from MIS can be seen in Figure 2(b), where G_0, G_1, etc. represent graphs on different levels of the pyramid. Note that we remove parallel edges and self-loops that emerge from the contractions, if they are not needed to encode inclusion of regions by other regions (in the example of Figure 2(b) we do not need loops nor parallel edges). This can be done by the dual graph contraction algorithm [Kro95].

2.2 Experiments with MIS

Uniformly distributed random numbers are given to vertices in the base graphs. We generated 1000 graphs, on top of which we built stochastic graph pyramids. In our experiments we used graphs of sizes 10000 and 40000 nodes, which correspond to image sizes 100×100 and 200×200 pixels, respectively. Figure 3 summarizes result of the first 100 of 1000 tests. Data in Table 1 were derived using graphs of size 200×200 nodes with 1000 experiments. We extract these parameters, the height of the pyramid, the maximum and the mean of the degree of vertices [2], and the number of iteration for correction to complete maximal independent set for any graph in the contraction process. We average these values on the whole data set. The degree of the vertex is of importance because directly related to the memory costs for the graph's representation [Jol02].

The number of levels needed to reduce the graph at the base level (level 0) to an apex (top of the pyramid) are given in Figure 3(a),(b). The vertical axis indicates the number of nodes on the levels indicated by the horizontal axis. The slopes of the lines correspond to the reduction factors. From Figure 3(a),(b) we see that the height of the pyramid cannot be guaranteed to be logarithmic, except for some good cases. In the worst case the pyramid had 22 levels for

[2] the number of edges incident to a vertex

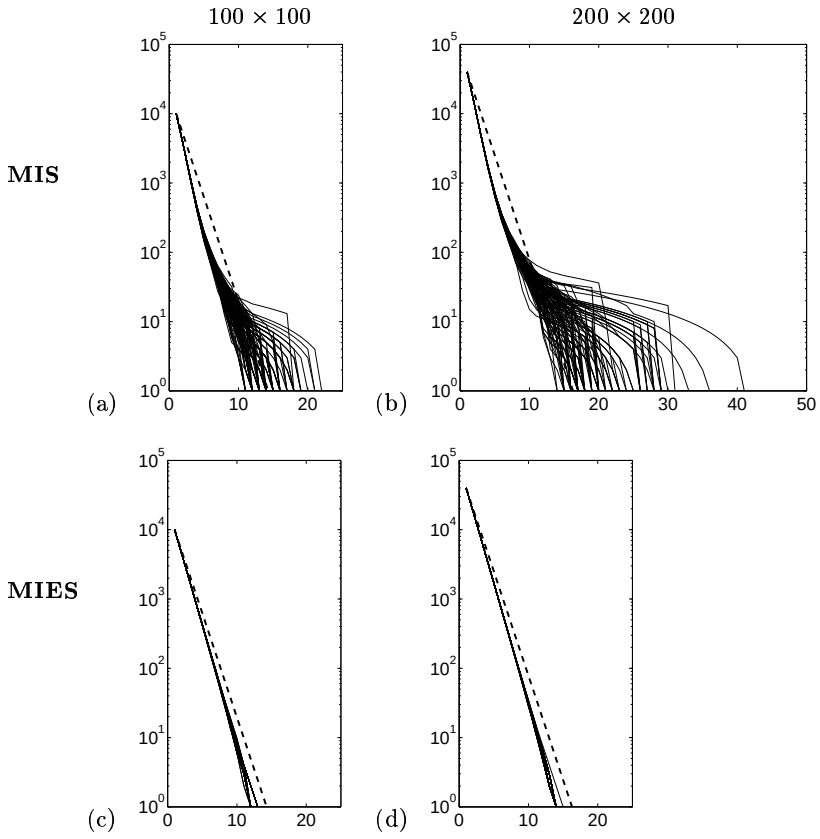

Fig. 3. Comparing MIS and MIES. Number of vertices in levels of MIS and MIES pyramids. The base levels are rectangular grid graphs containing 100×100 and 200×200 vertices . Dashed lines represent the theoretical reduction factor of 2.0.

the 100×100, respectively 41 levels for the 200×200 node graph. In these cases we have a very poor reduction factor. A poor reduction factor is likely, as can be seen in Figure 3(a),(b), especially when the images are large. This is due to the evolution of larger and larger variations between the vertex degrees in the contracted graphs (Table 1). The absolute maximum vertex degree was 148. The *a priori* probability of a vertex being the local maximum dependents on its neighborhood. The larger the neighborhood the smaller is the *a priori* probability that a vertex will survive. The number of iterations necessary for correction are the same as reported by [Mee89](Table 1 $\mu(\#iter) = 2.95$)).

To summarize, a constant reduction factor cannot be guaranteed.

3 Maximum Independent Edge Set

In the following we aim at a collection $\mathcal{C}$ of contraction kernels in a plane graph $\overline{G}$ such that

- each vertex of $\overline{G}$ is contained in exactly one kernel of $\mathcal{C}$, and
- each kernel $\mathcal{C}$ contains at least two vertices.

The contraction of all kernels in $\mathcal{C}$ will reduce the number of vertices to half or less.

3.1 Maximum Independent Edge Set Algorithm (MIES)

We start with independent *edge* sets or *matchings*, i.e. edge sets in which no pair of edges has a common end vertex. The maximal independent edge set (MIES), $\mathcal{C}$ is done in three steps.

1. Find a maximal matching M of edges in $\overline{G}$.
2. M is enlarged to a set M^+ that induces a spanning subgraph of $\overline{G}$.
3. M^+ is reduced to a subset defining $\mathcal{C}$.

A maximal matching of $\overline{G}$ is equivalent to a maximal independent vertex set on the edge graph of $\overline{G}$ [Die97,Chr75] . Thus, a maximal matching may be determined by the iterated local process as used in MIS algorithm.

Note that M is only required to be maximal, i.e. the edge set M cannot be enlarged by another edge from $\overline{G}$ without loosing independence. A maximal matching M is not necessarily maximum: there may be a matching M' (Figure 4(b)) that contains more edges than M (Figure 4(a)). The collection of contraction kernels defined by a maximal matching M may include kernels with a single vertex. Let v denote such an isolated vertex (isolated from M) and choose a non-self-loop e that has v as an end vertex. Since M is maximal, the end vertex $w \neq v$ of e belongs to an edge that is contained in the matching. Let M^+ denote the set of edges that are in M or that are chosen to connect isolated vertices to M.

The subgraph of $\overline{G}$ that is induced by M^+ spans $\overline{G}$ and its connected components are trees of depth one or two (Figure 4(c)). In the second case, the tree can be separated in two trees of depths one by removing the central edge $\in M$, indicated by the crosses in Figure 4(c). Still, each vertex of $\overline{G}$ belongs to a tree, now of depth one. The arrows in Figure 4(d) indicate possible directions of contractions. Since each vertex of $\overline{G}$ is now contained in a non-trivial contraction kernel, we proved the following.

Proposition 1 (Reduction factor of MIES at least 2.0). *The MIES algorithm yields contraction kernels for parallel contractions with a reduction factor of at least* 2.0.

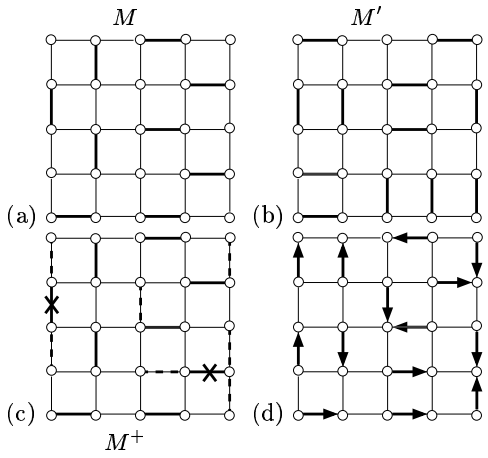

Fig. 4. (a) M: a maximal matching. (b) M': a matching with more edges than in (a). (c) M^+: the matching from (a) enlarged by connecting formerly isolated vertices to the maximal matching. (d) trees of depth two split into trees of depth one.

Note that in case of kernels with more than one edge the directions within the kernel cannot be chosen independently of one another. This is why the proposed method cannot be extended to applications in which there are a priori constraints on the directions of the contractions. However, the proposed method works for the stochastic case (no preconditions on edges to be contracted) and for connected component analysis, where the attributes of the end vertices are required to be identical.

3.2 Experiments with MIES

The same set of 1000 graphs was used to test MIES. The numbers of levels needed to reduce the graph on the base level to an apex of the pyramid are shown in Figure 3 (c),(d). Again the vertical axis indicates the number of vertices in the levels indicated by the horizontal axis. The experiments show that the reduction factor of MIES is indeed never smaller than the theoretical lower bound 2.0 (indicated by the dashed line in Figure 3(c),(d). MIES is more stable than MIS, as can be seen in Figure 3(c),(d) the variance of the slopes is smaller than in case of MIS, Figure 3(a),(b).

In Table 1 are given the height of the pyramid; the maximum vertex degree; the mean of vertex degree; and the number of iteration for correction averaged over the whole data set. The μ and σ of the height of the pyramid is smaller for MIES than for MIS. The same observation is also for maximal degree of vertices. The number of iterations for correction is higher for MIES (4.06) than for MIS (2.95). The two methods lead to almost the same mean vertex degree. The dependence of the data on this value seems to be little in both cases.

Table 1. Comparison of MIS and MIES.

Process	$\mu(height)$	$\sigma(height)$	$\mu(max)$	$\sigma(max)$	$\mu(mean)$	$\sigma(mean)$	$\mu(\#iter)$	$\sigma(\#iter)$
MIS	20.78	5.13	70.69	23.88	4.84	0.23	2.95	0.81
MIES	14.01	0.10	11.74	0.71	4.78	0.45	4.06	1.17

4 Conclusion and Outlook

Experiments with (stochastic) irregular image pyramids using maximal independent vertex sets (MIS) showed that reduction factor could not be bound as the image get larger. After an initial phase of strong reduction, the reduction decreases dramatically. This is due to the evolution of larger and larger variations between the vertex degrees in the contracted graphs. To overcome this problem we proposed a method (MIES) based on matchings which guarantees a reduction factor of 2.0. As in the case of independent vertex sets, the method based on matchings constrains the directions of the contractions. Future work will focus on improving the reduction factors also for the case of directional constraints i.e. directed edges. First experiments with a modification of the algorithm that addresses these constraints show comparable results to MIES.

References

[BCR90] M. Bister, J. Cornelis, and Azriel Rosenfeld. A critical view of pyramid segmentation algorithms. *Pattern Recognition Letters*, 11(9):605–617, 1990.

[Chr75] N. Christofides. *Graph theory - an algorithmic approach.* Academic Press, New York, 1975.

[Die97] Reinhard Diestel. *Graph Theory.* Springer, New York, 1997.

[JM92] Jean-Michel Jolion and Annick Montanvert. The adaptive pyramid, a framework for 2D image analysis. *Computer Vision, Graphics, and Image Processing: Image Understanding*, 55(3):pp.339–348, May 1992.

[Jol02] Jean-Michel Jolion. Stochastic pyramid revisited. *Pattern Recognition Letters. To appear*, 2002.

[JR94] J.M. Jolion and Azriel Rosenfeld. *A Pyramid Framework for Early Vision.* Kluwer Academic Publishers, Dordrecht, Netherlands, 1994.

[KLB99] Walter G. Kropatsch, Aleš Leonardis, and Horst Bischof. Hierarchical, Adaptive and Robust Methods for Image Understanding. *Surveys on Mathematics for Industry*, No.9:1–47, 1999.

[KM95] Walter G. Kropatsch and Herwig Macho. Finding the structure of connected components using dual irregular pyramids. In *Cinquième Colloque DGCI*, pages 147–158. LLAIC1, Université d'Auvergne, September 1995.

[Kro95] Walter G. Kropatsch. Building Irregular Pyramids by Dual Graph Contraction. *IEE-Proc. Vision, Image and Signal Processing*, 142(6):366 – 374, 1995.

[Mee89] Peter Meer. Stochastic image pyramids. *CVGIP*, 45:269 – 294, 1989.

[MMR91] Annick Montanvert, Peter Meer, and Azriel Rosenfeld. Hierarchical image analysis using irregular tesselations. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(4):pp.307–316, April 1991.