Constructing Stochastic Pyramids by MIDES - Maximum Independent Directed Edge Set *

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Abstract. We present a new method to determine contraction kernels for the construction of graph pyramids. Experimentally the new method has a reduction factor higher than 2.0. Thus, the new method yields a higher reduction factors than the stochastic decimation algorithm (MIS) and maximal independent edge set (MIES), in all tests. This means the number of vertices in the subgraph induced by any set of contractible edges is reduced to half or less by a single parallel contraction. The lower bound of the reduction factor becomes crucial with large images.

Keywords.Irregular graph pyramids, maximal independent set, maximal independent edge set, topology preserving contraction.

1 Introduction

In a regular image pyramid (for an overview see [16]) the number of pixels at any level k, is r times higher than the number of pixels at the next reduced level $k+1$. The so called reduction factor r is greater than 1 and it is the same for all levels k. If P denotes the number of pixels in an image I, the number of new levels on top of I amounts to $log_r(P)$ (Figure 1(a)). Thus, the regular image pyramid may be an efficient structure to access image objects in a top-down process. For more in depth on the subject see the book of Rosenfeld [20].

However, regular image pyramids are confined to globally defined sampling grids and lack shift invariance [2]. In [19, 10] it was shown how these drawbacks can be avoided by irregular image pyramids, the so called adaptive pyramids. Each level represents a partition of the pixel set into cells, i.e. connected subsets of pixels. The construction of an irregular image pyramid is iteratively local [18, 9, 1]:

* This paper has been supported by the Austrian Science Fund under grants P14445-MAT and P14662-INF

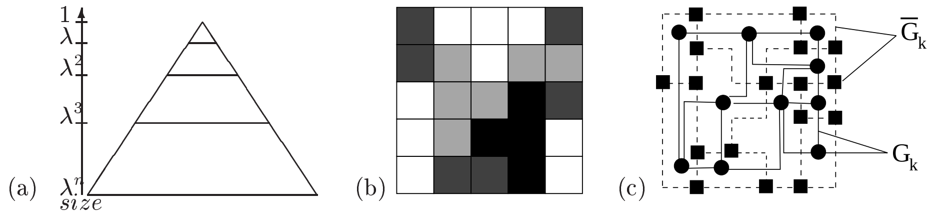

Fig. 1. (c) Pyramid concept. (a) Partition of pixel set into cells. (b) Representation of the cells and their neighborhood relations by a dual pair $(G, \overline{G})$ of plane graphs.

- the cells have no information about their global position.
- the cells are connected only to (direct) neighbors.
- the cells cannot distinguish the spatial positions of the neighbors.

On the base level (level 0) of an irregular image pyramid the cells represent single pixels and the neighborhood of the cells is defined by the 4-connectivity of the pixels. A cell on level $k+1$ (parent) is the union of some neighboring cells on level k (children). This union is controlled by so called contraction kernels (decimation parameters [13]). This implies that an image pyramid is built in $O[log(image\_diameter)]$ parallel steps [1]. For more in depth on the subject see the book of Jolion [11]. Neighborhoods on level $k+1$, are derived from neighborhoods on level k. Two cells c_1 and c_2 on level $k+1$ are neighbors if there exist children p_1 of c_1 and p_2 of c_2 such that p_1 and p_2 are neighbors in level k, Figure 1(b). We assume that on each level $l+1$ ($l \geq 0$) there exists at least one cell not contained in level l. In particular, there exists a highest level h . Furthermore, we restrict ourselves to irregular pyramids with an apex, i.e. level h contains one cell.

In this paper we represent the levels as dual pairs $(G_k, \overline{G_k})$ of plane graphs G_k and $\overline{G_k}$, Figure 1(c). The vertices of G_k represent the cells and the edges of G_k represent the neighborhood relations of the cells on level k, depicted with circle vertices and solid edges in Figure 1(c). The edges of $\overline{G_k}$ represent the borders of the cells on level k, depicted with dashed lines in Figure 1(c), possibly including so called pseudo edges needed to represent the neighborhood relation to a cell completely surrounded by another cell. Finally, the vertices of $\overline{G_k}$, the squares in Figure 1(c), represent points where at least three edges from $\overline{G_k}$ meet. The sequence $(G_k, \overline{G_k})$, $0 \leq k \leq h$ is called (dual) graph pyramid.

The homogeneous region does not offer additional information to be used to contract this region. Thus the stochastic selection principle is used to contract a homogeneous region. The aim of this paper is to combine the advantage of regular pyramids (logarithmic tapering) with the advantages of irregular graph pyramids (their purely local construction and shift invariance). The aim is reached by exchanging the stochastic selection method (MIS) for contraction kernels proposed in [18] by another iteratively local method (MIDES) that has a reduction factor

[1] For images with $P = A * B$ pixel, $B = A * a$, $image\_diameter = \sqrt{A^2 + B^2}$ yields $O[log(P)] = O[log(image\_diameter)]$

higher than 2.0. The other goal is not to be limited by the direction of contraction, which is a drawback of MIES [7]. Experiments with both selection methods show that:

- the MIS method does not lead necessarily to logarithmic tapering graph pyramids, i.e. the reduction factors of graph pyramids built by the MIS can get arbitrarily close to 1.0 [19].
- the sizes of the receptive fields from the new method (MIDES) are much more uniform.
- MIES is limited to the cases where the direction of contraction is not important.

Not only stochastic decimation [18], but also connected component analysis [17] gains from the new method.

The plan of the paper is as follows. In Section 2 we recall the main idea of the stochastic pyramid algorithm and in Section 2.3 we see that graph pyramids from maximal independent vertex sets (MIS) may have a very small reduction factor. Moreover, experiments show that small reduction factors are likely, especially when the images are large. We propose a new method (MIDES), in Section 3, based on directed graphs and show in Section 3.2 that this method has a reduction factor higher than 2.0.

2 Maximal Independent Vertex Set

In the following the iterated stochastic construction of the irregular image pyramid in [18] is described in the language of graph pyramids. The main idea is to first calculate a so called *maximal independent vertex set*[2] [4]. Let V_k and E_k denote the vertex set and the edge set of G_k, respectively and let $\iota(\cdot)$ be the mapping from an edge to its set of end vertices. The neighborhood $\Gamma_k(v)$ of a vertex $v \in V_k$ is defined by

$$\Gamma_k(v) = \{v\} \cup \{w \in V_k \mid \exists e \in E_k \text{ such that } v, w \in \iota(e)\}.$$

A subset W_k of V_k is called maximal independent vertex set if:

1. $w_1 \notin \Gamma_k(w_2)$ for all $w_1, w_2 \in W_k$,
2. for all $v \in V_k \setminus W_k$ there exists $w \in W_k$ such that $v \in \Gamma_k(w)$.

Put in words, two members (survivors) of maximal independent vertex set cannot be neighbors (condition 1) and every non-member (non-survivor) is in the neighborhood of at least one member (condition 2). An example of a maximal independent vertex set is shown with black vertices in Figure 2(a), the arrows indicate a corresponding collection of contraction kernels.

[2] also called maximal stable set; we distinct maximal from maximum independent set, which construction is NP-complete. See [4, 21] for algorithmic complexity.

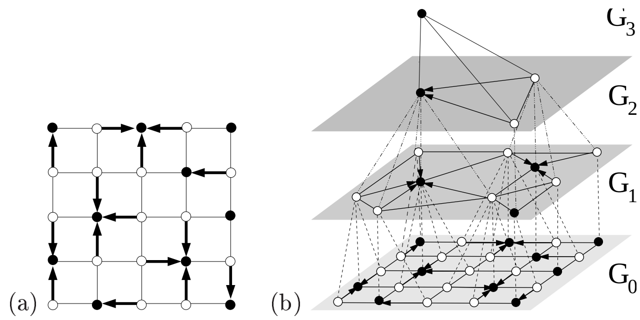

Fig. 2. (a) Maximal independent vertex set. (b) A graph pyramid from maximal independent vertex set.

2.1 Maximal Independent Vertex Set Algorithm (MIS)

The maximal independent vertex set (MIS) problem was solved using a heuristic in [18]. The number of iterations to complete maximal independent set converges in most of the cases very fast, only few iterations for correction [18] are needed. MIS may be generated as follows.

Algorithm 1 – MIS Algorithm

1: Mark every vertex v of V_l as *candidate*.
2: **while** there are candidates **do**
3: Assign random numbers to the candidates of V_l.
4: Determine the candidates whose random numbers are larger than the random numbers of all neighboring candidates and mark them as *member* (of the maximal independent set) and as *non-candidate*. Also mark every neighbor of every new member as *non-candidate* and as *non-member*.
5: **end while**
6: In each neighborhood of a vertex that is not a member there will now be a member. Let each non-member choose its neighboring member, say the one with the maximal random number (we assume that no two random numbers are equal).

The assignment of the non-members to their members determines a collection of *contraction kernels*: each non-member is contracted toward its member and all contractions can be done in a single parallel step. In Figure 2(a) the contractions are indicated by arrows. A stochastic graph pyramid with MIS can be seen in Figure 2(b), where $G_0, G_1, ...$ etc. represent graphs on different levels of the pyramid. Note that we remove parallel edges and self-loops that emerge from the contractions, if they are not needed to encode inclusion of regions by other regions (in the example of Figure 2(b) we do not need loops nor parallel edges). This can be done by the dual graph contraction algorithm [14].

2.2 Experimental Setup and Evaluation

Uniformly distributed random numbers are assigned to vertices or edges in the base grid graphs. We generated 1000 graphs, on top of which we build stochastic graph pyramids. In our experiments we use graphs of sizes 10000 and 40000 nodes, which correspond to image sizes 100×100 and 200×200 pixels, respectively. Figure 3 summarizes the result of the first 100 of 1000 tests. Data in Table 1 and Table 2 were derived using graphs of size 200×200 nodes with 1000 experiments.

We extract the following parameters: the height of the pyramid (*height*); the maximum and the mean of the degree of vertices [3] (*max-degree* and *mean-degree*); and the number of iterations for correction (*correction*) to complete maximal independent set for any graph in the contraction process (Table 1). In Table 2 are shown reduction factors for vertices ($|V_k|/|V_{k+1}|$) and edges ($|E_k|/|E_{k+1}|$). We average these values on the whole data set (μ-mean and σ-standard deviation). The degree of the vertex is of importance because it is directly related to the memory costs for the graph's representation [9]. Numerical values used for drawing Figure 3 are given in [8]. We compare the quality of selection methods in Section 2.3 and 3.2.

2.3 Experiments with MIS

The number of levels needed to reduce the graph at the base level (level 0) to an apex (top of the pyramid) are given in Figure 3(a),(b). The vertical axis indicates the number of nodes on the levels indicated by the horizontal axis. The slopes of the lines correspond to the reduction factors. From Figure 3(a),(b) we see that the **height** of the pyramid **cannot be guaranteed to be logarithmic**, except for some good cases. In the worst case the pyramid had 22 levels for the 100×100, respectively 41 levels for the 200×200 graphs. See [8] for numerical details. In these cases we have a very poor reduction factor. A **poor reduction factor** is likely, as can be seen in Figure 3(a),(b), especially when the **images are large**. This is due to the evolution of larger and larger variations between the vertex degrees in the contracted graphs (Table 1 *max-degree* and *mean-degree* columns). The absolute maximum vertex degree was 148. The *a priori* probability of a vertex being the local maximum dependents on its neighborhood. The larger the neighborhood the smaller is the *a priori* probability that a vertex will survive. The number of iterations necessary for correction are the same as reported by [18] (Table 1 *correction* columns).

To summarize, a **constant reduction factor cannot be guaranteed**.

3 Maximum Independent Directed Edge Set

In many graph pyramid applications such as line image analysis [3, 15], the description of image structure [5, 6, 12], a directed edge e with source u and target

[3] the number of edges incident to a vertex, i.e the number of non survivor contracted into the survivor.

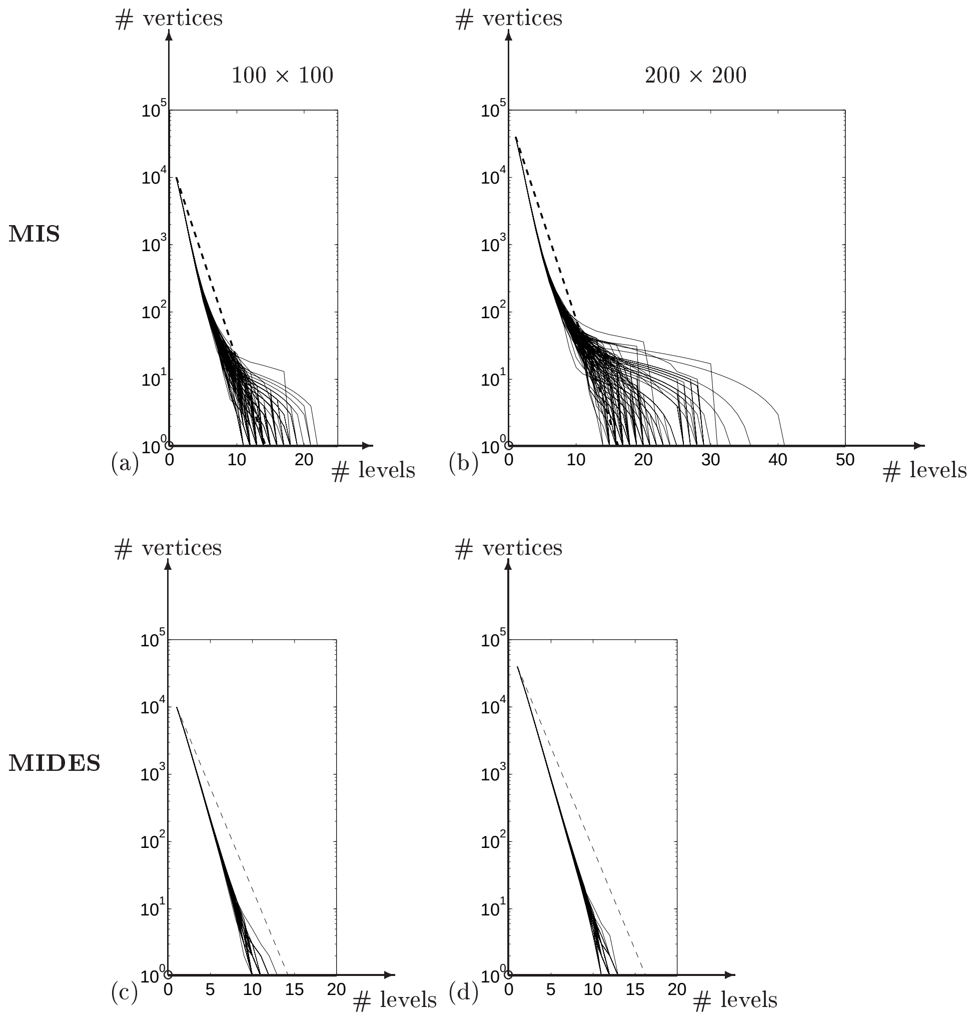

Fig. 3. Comparing MIS and MIDES. Number of vertices in levels of MIS and MIDES pyramids. The base levels are rectangular grid graphs containing 100×100 and 200×200 vertices . Dashed lines represent the theoretical reduction factor of 2.0 (MIES [7]).

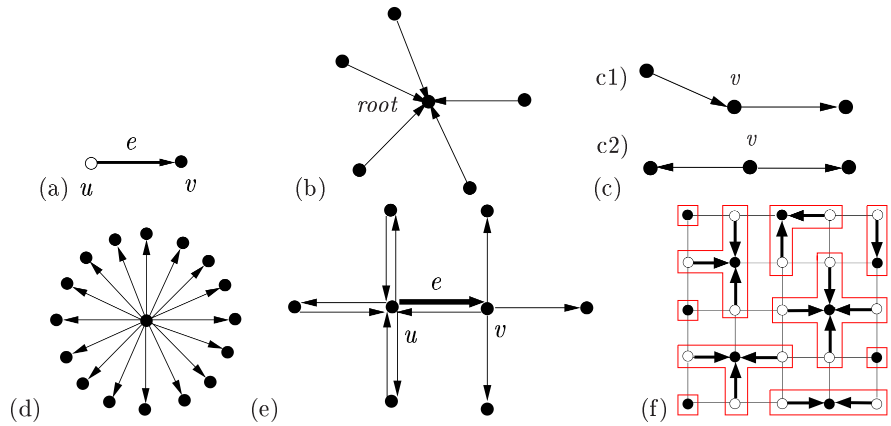

Fig. 4. (a) The direction of contraction. (b) A legal configuration of directed edges. (c) Forbidden pairs of directed edges. (d) The reduction factor of a star with n edges pointing away from the center is $n/(n-1)$. (e) The directed neighborhood $N(e)$. (f) Maximal independent directed edge set with respect to $N(e)$.

$v \neq u$ must be contracted (from u to v, Figure 4(a)), only if the attributes of e, u, and v fulfill a certain condition, making the direction of contraction an important issue. In line drawings end point of lines or intersection must be preserved for geometric accuracy reasons. In particular, the condition depends on u being the source and v being the target. The edges that fulfill the condition are called *preselected* edges.

From now on the plane graphs in the pyramid have (bi)directed edges. Typically, the edges in the base level of the pyramid form pairs of reverse edges, i.e. for each edge e with source u and target v there exists an edge e' with source v and target u. However, the set of preselected edges may contain e without containing e'. The goal is to build contraction kernels with a "high" reduction factor from the set of preselected edges, such that a predetermined target v survives. The reduction will always be determined according to the directed graph induced by the preselected edges. For example, if the number of vertices in the induced subgraph is reduced to half, the reduction factor will be 2.0.

Definition 1. *A contraction kernel is a vertex disjoint rooted tree of depth one or zero (single vertex), each edge of which is directed toward the root (survivor).*

Note that the direction of edges uniquely determines which vertex survives on the next level of the pyramid, i.e determines the contraction kernel (decimation parameter [13]). Figure 4(b) shows a configuration of directed edges, which yields a contraction kernel.

Definition 2. *Let v be a vertex of bi-directed graph G than,*

$$s(v) := \#\{e \in G \;|v \text{ is source of } e\}.$$

$$t(v) := \#\{e \in G \; | v \text{ is target of } e\}.$$

In Figure 4(b) the number of edges with target in *root* is $t(root) = 5$; and for the center vertex with n edges pointing away $s(center) = n$, Figure 4(d).

Proposition 1. *Let D denote a set of directed edges from a bi-directed graph G and G_D denote the subgraph induced by D. Then the following statements are equivalent:*

(a) $s(v) < 2 \;\wedge\; s(v) \cdot t(v) = 0, \quad \forall v \in G_D$.
(b) *G_D is a vertex disjoint union of contraction kernels.*

Proof. **i)** (a) $\Rightarrow$ (b): Let $R := \{r : \; r$ is target of some $e \in D \wedge s(r) = 0\}$ be the set of roots. Furthermore, set $E_r := \{e \in D : \; r$ is target of e$\}$, $r \in R$. Then $E = \bigcup_{r \in R} E_r$ and E_r induces a contraction kernel C_r for any $r \in R$. It remains to show that the C_r are vertex disjoint. Assume the opposite, i.e. there exists a vertex u contained in C_v and in C_w for some $v \neq w \in R$. From (a) it follows that $u \notin \{v, w\}$. Hence, there exist edges $(u, v), (u, w) \in D$, a contradiction to $s(u) < 2$.

ii) (b) $\Rightarrow$ (a): Let T be the set of roots of the vertex disjoint contraction kernels and let C_t denote the unique contraction kernel with root t, $t \in T$. Furthermore, let $v \in G_D$. Since the C_t are vertex disjoint, exactly one of the following holds:
1. $v \in T$ and $s(v) = 0$.
2. $v \notin T$ and $s(v) = 1, t(v) = 0$.

Kernels which do not fulfill the Definition 1 are forbidden kernels. Forbidden kernels do not allow certain pair of edges as shown in Figure 4(c1) where $s(v) = 1$ and $t(v) = 1$; and (c2) where $s(v) = 0$ and $t(v) = 2$. In general forbidden kernels have one or both of the cases shown in Figure 4(c). A set C of directed edges forms a collection of contraction kernels if C fulfill the Proposition 1 i.e. contains no forbidden edge pairs. From the example in Figure 4(d) it is clear that only one edge can be contracted (otherwise one ends with forbidden contraction kernels), which means in general, vertex reduction factor can get arbitrarily close 1.0.

Note that, in contrast to MIS, the roots of two contraction kernels may be neighbors. If a preselected directed edge $e = (u, v)$ belongs to a contraction kernel, the edges violating condition a) in Proposition 1 form the set $N(e)$ defined below.

Definition 3. *Let $e = (u, v)$ be a directed edge of G. Then the directed neighborhood $N(e)$ is given by all directed edges $e' = (u', v')$ such that*

$$u \in \{u'\} \cup \{v'\} \vee u' = v.$$

Neighborhood $N(e)$ of e is given by edges which point toward the source of e, edges with the same source u as e and the edges the source of which is the target of e. Note that edges pointing towards the target of e are not a part of the directed neighborhood. Figure 4(e) depicts $N(e)$ in case of u and v both having 4 neighbors. A method to find a set of contraction kernels C, which is similar to the MIES method [7], but now with directed edges (taking into the consideration the direction of contraction) is given in the section below.

3.1 Maximal Independent Directed Edge Set Algorithm (MIDES)

To find a **maximal** (independent) set of directed edges (MIDES) forming vertex disjoint rooted trees of depth zero or one, we proceed analogously to the generation of maximal independent vertex sets (MIS), as explained in the Section 2. Let E_k denote the set of bi-directed edges in the graph G_k of the graph pyramid. We proceed as follows.

Algorithm 2 – **MIDES** Algorithm

1: Mark every directed edge e of E_k as *candidate*.
2: **while** there are candidates **do**
3: Assign random numbers to the candidates of E_k.
4: Determine the candidates e whose random numbers are larger than the random numbers of candidates in its directed neighborhood $N(e) \setminus \{e\}$ and mark them as *member* (of a contraction kernel) and as *non-candidate*. Also mark every $e' \in N(e)$ of every new member e as *non-candidate*.
5: **end while**

Since the direction of edges uniquely determines the roots of the contraction kernels (the survivors), all the vertices which are the sources of directed edges are marked as non-survivor. An example of a set of contraction kernels C found by MIDES is given in Figure 4(f) (the survivors are depicted with black and non-survivors with white). Since we are using a maximal independent set of directed edges, similarly to MIS we cannot guarantee the constant reduction factor.

3.2 Experiments with Maximal Independent Directed Edge Sets

The same set of 1000 graphs (Section 2.2) was used to test MIDES. The numbers of levels needed to reduce the graph on the base level to an apex of the pyramid are shown in Figure 3 (c),(d). Again the vertical axis indicates the number of vertices in the levels indicated by the horizontal axis. The experiments show that the **reduction factor** of MIDES is indeed **never smaller** than the lower bound 2.0 (indicated by the dashed line in Figure 3(c),(d), even in the worst case (*height* 13 see [8] for details on numerical results). Also the *in-degrees* of the vertices is much smaller than for MIS (*max-degree* column in Table 1). For the case of the graph with size 200×200 vertices, MIDES needs less levels than MIS and MIES and the number of iterations needed to complete the maximum independent set was comparable with the one of MIS (Table 1, *correction* column). In Table 2, statistics of reduction factors for vertices ($|V_k|/|V_{k+1}|$) and edges ($|E_k|/|E_{k+1}|$) are given. From this table one can read that **MIDES** algorithm shows a **better reduction factor** than MIES [7] and MIS.

To summarize the experiments show that **MIDES** has a **reduction factor upper bounded by** 2, even though we did not prove theoretically this upper bound, as it was done for MIES in [7]. Nevertheless this method shows the best reduction factor of MIS or MIES.

Table 1. Comparison of MIS,MIES and MIDES.

Process	*height* μ	*height* σ	*max-degree* μ	*max-degree* σ	*mean-degree* μ	*mean-degree* σ	*correction* μ	*correction* σ
MIS	20.78	5.13	70.69	23.88	4.84	0.23	2.95	0.81
MIES	14.01	0.10	11.74	0.71	4.78	0.07	4.06	1.17
MIDES	12.07	0.46	13.29	1.06	4.68	0.14	2.82	1.07

Table 2. Reduction factors of vertices and edges.

Process	$\frac{\lvert V_k\rvert}{\lvert V_{k+1}\rvert}$ μ	$\frac{\lvert V_k\rvert}{\lvert V_{k+1}\rvert}$ σ	$\frac{\lvert E_k\rvert}{\lvert E_{k+1}\rvert}$ μ	$\frac{\lvert E_k\rvert}{\lvert E_{k+1}\rvert}$ σ
MIS	1.94	1.49	1.78	0.69
MIES	2.27	0.21	2.57	1.21
MIDES	2.62	0.36	3.09	1.41

4 Conclusion and Outlook

Experiments with (stochastic) irregular image pyramids using maximal independent vertex sets (MIS) showed that the reduction factor can get arbitrarily close to 1.0 for large images. After an initial phase of strong reduction, the reduction decreases dramatically. This is due to the evolution of larger and larger variations between the vertex degrees in the contracted graphs. To overcome this problem we proposed a new method (MIDES). This method has a reduction factor higher than 2.0, which was experimentally tested. MIDES is not constrained by the directions of the contractions as in the case of MIES [7].

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