Irregular Graph Pyramids, Integral Operators and Representative Cocycles of Cohomology Generators*

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Abstract This paper introduces an algorithm to efficiently compute representative cocycles (the basic elements of cohomology) in the boundary graph of any level of a given irregular graph pyramid. Extention to nD and future topics of interest are discussed.

Keywords: irregular graph pyramids, representative cocycles of cohomology generators

1 Introduction

A region adjacency graph (RAG) is defined for a partition by associating a vertex to each region and by creating an edge between two vertices if the associated regions share a common boundary i.e. if they are adjacent. Considering the RAG associated to a 2D binary digital image, the number of maximal connected subgraphs having only vertices labeled as foreground is equal to the number of independent pieces of the image (connected components of the foreground). Even if the embedding of the RAG is known, computing the number of holes of the object is not straightforward. One way would be to consider the cycles with exactly 4 edges as *degenerate cycles* and establish an equivalence between all the cycles of the graph as follows: two cycles are equivalent if one can be obtained from the other by joining to it one or more degenerate cycles. There is only one equivalence class for the set of black pixels of the digital image of Figure 1, which represents the unique hole. This is similar to consider the digital image as a cell complex[1] [5]. One can ask for the edges we have to delete in order to 'destroy' the hole. In the example in Figure 1 it is not enough to delete only one edge. The set of edges in blue in Figure 1 block any cycle that surrounds the hole; the deletion of these edges together with the faces that they bound produces the 'disappearing' of the hole. A 1-*cocycle* of a planar object can be seen as a set of edges 'blocking' the creation of cycles of one homology class. The elements of *cohomology* are equivalence classes of cocycles.

2 Irregular Graph Pyramids

Irregular graph pyramids [8, 9] are used within the segmentation framework to encode a hierarchy of partitions. An irregular graph pyramid P is a stack of successively reduced planar graphs

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[1] Intuitively a cell complex is defined by a set of 0-cells (vertices) that bound a set of 1-cells (edges), that bound a set of 2-cells (faces), etc.

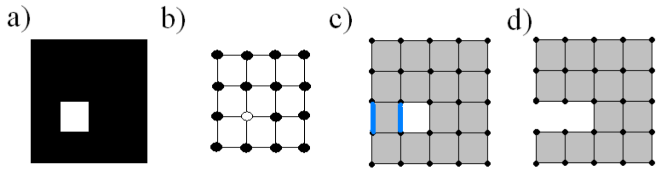

Figure 1: a) A 2D binary digital image I; b) its region adjacency graph; c) a cell complex associated to I (in blue, a representative cocycle); and d) the cell complex without the hole.

$P = \{(G_0, \bar{G}_0), \ldots, (G_k, \bar{G}_k)\}$ where G_i are *adjacency graphs* and $\bar{G}_i$ are *boundary graphs* (their dual), for $0 \leq i \leq k$. The vertices of G_i represent the image regions on level i and its edges represent the neighborhood relations. The edges of $\bar{G}_i$ represent the borders of the regions on level i, including so called *pseudo edges* needed to represent neighborhood relations to a region completely enclosed by other regions. Finally, the vertices of $\bar{G}_i$ represent junctions of border segments of $\bar{G}_i$.

In each level i, there exists a one-to-one correspondence between the edges of the primal and the edges of the dual. This induces a one-to-one correspondences between the vertices of the primal and the 2D cells obtained from the embedding of the dual graph (will be denoted by the term *faces*[2]): if two vertices are connected by an edge, then the two corresponding faces of the dual share an edge. Note that both G_i and $\bar{G}_i$ can play the role of primal and dual. Each $\bar{G}_i$ is obtained by first removing edges of $\bar{G}_{i-1}$ if they are in the common boundary of two faces with the same label, and then contracting edges until all vertices have degree greater than 2. See Figure 2. Note that a contraction in the primal is equivalent to a removal in the dual, and vice-versa. In all graphs of level $0 \leq i < k$, contracted edges define trees called *contraction kernels*, whose vertices are merged to a single vertex. One vertex of each contraction kernel is called the *surviving vertex* and is considered to have been survived to the next level. For each boundary graph $\bar{G}_i$, the cell complex associated to the foreground object, called *boundary cell complex*, is obtained by taking all faces corresponding to vertices of G_i whose receptive fields contain (only) foreground pixels, and adding all edges and vertices needed to represent the faces.

Lemma 1 *All the boundary cell complexes of a given irregular graph pyramid are cell subdivisions of the same object. Therefore, all these cell complexes are homeomorphic.*

3 Cohomology and Integral Operators

Starting from a cell decomposition of an object, its homology can be defined in an algebraic way by studying incidence relations of its subdivision. A cell of dimension p is called a p-cell. A *p-chain* is a formal sum of p-cells. The chains are considered over $\mathbf{Z}/2$ coefficients. The set of p-chains form an abelian group called the *p-chain group* and denoted by C_p. This group is generated by all

[2]Not to be confused with the vertices of the dual of a RAG (sometimes also denoted by the same term).

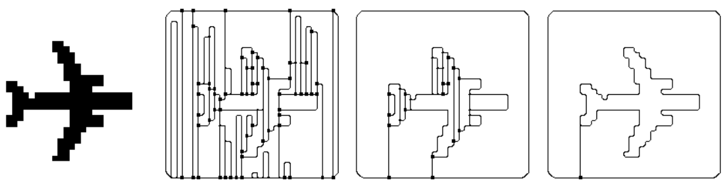

Figure 2: A digital image I, the boundary graphs of levels 6, 10, and the last one of an irregular graph pyramid of I.

the p-cells. The *boundary operators* are a set of homomorphisms $\{\partial_p : C_p \rightarrow C_{p-1}\}_{p\geq 0}$ connecting two immediate dimensions: $\cdots \stackrel{\partial_{p+1}}{\rightarrow} C_p \stackrel{\partial_p}{\rightarrow} C_{p-1} \rightarrow \cdots \stackrel{\partial_1}{\rightarrow} C_0 \stackrel{\partial_0}{\rightarrow} 0$.

By linearity, the boundary of any p-chain is the sum of the boundaries of each cell that appears in the chain. The boundary of the 0-cells is always 0. For each p, $\partial_{p-1}\partial_p = 0$. A p-chain σ is called a *p-cycle* if $\partial_p(\sigma) = 0$. If $\sigma = \partial_{p+1}\mu$ for some $(p+1)$-chain μ then σ is called a *p-boundary*. Denote the groups of p-cycles and p-boundaries by Z_p and B_p respectively. It is true that $B_p \subseteq Z_p$. Define the pth *homology group* to be the quotient group $H_p = Z_p/B_p$, for all p.

Construct cohomology groups by turning chain groups into groups of homomorphisms and boundary operators into their dual homomorphisms. Define a *p-cochain* as a homomorphism $c : C_p \rightarrow \mathbf{Z}/2$, which is totally defined by the set of p-cell that evaluates to 1. The p-cochains form the set C^p which is a group. The boundary operator defines a dual homomorphism, the *coboundary operator* $\delta^p : C^p \rightarrow C^{p+1}$, such that $\delta^p(c) = c\partial_{p+1}$ for any p-cochain c. Since the coboundary operator runs in a direction opposite to the boundary operator, it raises the dimension. Its kernel is the group of *cocycles* and its image is the group of *coboundaries*. Two cocycles c and c' are *cohomologus* if $c + c'$ is a coboundary. The p-th *cohomology group* is defined as the quotient of p-cocycle modulo p-coboundary groups, $H^p = Z^p/B^p$, for all p. If the object is embedded in $\mathbf{R}^3$, then homology and cohomology groups are isomorphic. The elements from which we can deduce all H^p elements are called *cohomology generators* of dimension p. We say that c is a *representative p-cocycle* of a cohomology generator (or representative p-cocycle for short) β of dimension p if $\beta = c + B^p$. Denote $\beta = [c]$. See [10].

Starting from a cell decomposition of an object and the chain complex associated to it, $\cdots \stackrel{\partial_2}{\rightarrow} C_1 \stackrel{\partial_1}{\rightarrow} C_0 \stackrel{\partial_0}{\rightarrow} 0$, take a q-cell σ and a $(q+1)$-chain α. An *integral operator* [2] is defined as the set of homomorphisms $\{\phi_p : C_p \rightarrow C_{p+1}\}_{p\geq 0}$ such that $\phi_q(\sigma) = \alpha$ and is 0 if it is evaluated over all the other q-cells. It is extended to all q-chains by linearity. When $p \neq q$, then ϕ_p is the zero homomorphism. Integral operators can be seen as a kind of boundary operator inverse. They satisfy the condition $\phi_{p+1}\phi_p = 0$ for all p. If an integral operator $\{\phi_p : C_p \rightarrow C_{p+1}\}_{p\geq 0}$ satisfies the *chain-homotopy property*, that is, $\phi_p\partial_{p+1}\phi_p = \phi_p$ for each p, then define the set $\{\pi_p : C_p \rightarrow im\pi_p\}_{p\geq 0}$ where $\pi_p = id_p + \phi_{p-1}\partial_p + \partial_{p+1}\phi_p$ is a homomorphism, $\cdots \stackrel{\partial_2}{\rightarrow} im\pi_1 \stackrel{\partial_1}{\rightarrow} im\pi_0 \stackrel{\partial_0}{\rightarrow} 0$ is a chain complex and $\{id_p : C_p \rightarrow C_p\}_{p\geq 0}$ is the identity. Then π is a chain equivalence (its chain-homotopy inverse is the inclusion).

Lemma 2 *The two operations used to construct an irregular graph pyramid: edge removal and edge contraction, are integral operators satisfying the chain-homotopy property.*

Consider the cell complex K of Figure 3. The integral operator associated to the removal of the edge e is given by $\phi_1(e) = B$ and ϕ_p maps the other p-cells of K to 0, for $p = 0, 1, 2$. Then, $\pi_1(e) = a + f + d$, $\pi_2(B) = 0$, $\pi_2(A) = A + B$ (renamed as A' in K') and π_p is the identity over the other p-cells of K, $p = 0, 1, 2,$. Now, K'' is obtained from K' contracting the edges a and d. The set of homomorphism defined by $\phi'_0((1)) = a$, $\phi'_0((3)) = d$ and ϕ'_p over the other p-cells of K' is 0, $p = 0, 1, 2$, is a composition of two integral operators [2] representing these contractions. Then, $\pi'_0((1)) = (2)$, $\pi'_0((3)) = 4$, $\pi'_1(c) = c + a$ (renamed as c' in K''), $\pi'_1(d) = d + b$ (renamed as d' in K'') and π'_p is the identity over the other p-cells of K', for $p = 0, 1, 2$.

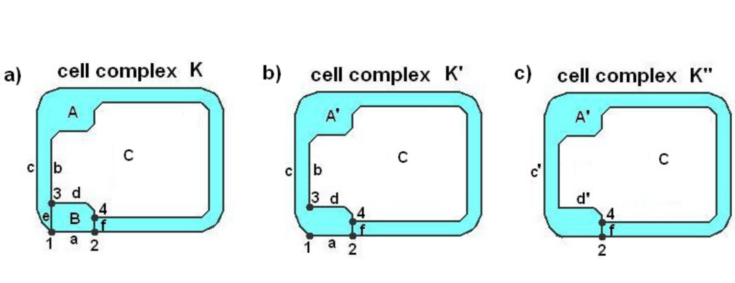

Figure 3: a) A cell complex K; b) K after removing e; and c) after contracting a and b.

Reconstructing the equivalent integral operators for a given graph pyramid is straight forward, as all information is contained in the horizontal (in-level) and vertical (between levels) relations.

Lemma 3 *Let $\{\phi_p : C_p \to C_{p+1}\}_{p\geq 0}$ be an integral operator satisfying the chain-homotopy property. The chain complexes $\cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$ and $\cdots \xrightarrow{\partial_2} im\pi_1 \xrightarrow{\partial_1} im\pi_0 \xrightarrow{\partial_0} 0$ have isomorphic cohomology groups. If $c : im\pi_p \to \mathbf{Z}/2$ is a representative p-cocycle of a cohomology generator, then $c\pi : C_p \to \mathbf{Z}/2$ is a representative p-cocycle of the same generator.*

For example, consider the cell complex K'' of Figure 3. The 1-cochain α, defined by the set $\{c', d'\}$ of edges of K'', is a 1-cocycle which 'represents' the white hole C (in the sense that all the cycles representing the hole must contain at least one of the edges of the set). Then $\beta = \alpha\pi'$ is defined by the set $\{c, d\}$ of edges of K'. Let $\gamma = \beta\pi$. Then γ is defined by the set $\{c, d, e\}$ of edges of K. β and γ are both 1-cocycles representing the white hole C.

4 Representative Cocycles on Irregular Graph Pyramids

In [6], the authors present a novel algorithm for correctly visualizing graph pyramids, including multiple edges and self loops, which preserves the geometry and the topology of the original image. A method for homology computation on irregular graph pyramids is given in [11].

In this paper, representative cocycles are computed and drawn in the boundary graph of any level of a given irregular graph pyramid. Representative cocycles are computed in the top (the last level) and down projected to the bottom (level 0) using the process described in this section. For this purpose, a new level, called *homology-generator level*, is added over the boundary graph of the last level of the pyramid. We compute a spanning tree of the boundary graph of the last level and contract the edges that belong to it, to a common point (the root of the tree). The boundary graph in the homology-generator level is a set of regions surrounded by a set of self loops incident in a common point. The composition of the integral operators associated with the contraction of edges of the spanning tree, defines a chain equivalence between the boundary cell complex of the last level of the pyramid and the boundary cell complex of the homology-generator level.

Lemma 4 *The boundary cell complex of any level of the pyramid and the one of the homology-generator level have isomorphic (co)homology groups.*

For example, in the boundary graph of the homology-generator level (see Figure 5) each self loop that surrounds a region of the background is a representative 1-cycle of a homology generator. In the same graph, the representative 1-cocycle of each cohomology generator is defined by exactly two self loops. One of them is the self loop α representing one homology generator, which also belongs to the boundary of a region R of the foreground of the homology-generator level. Let β be the self loop surrounding the region R. Then, $\{\alpha, \beta\}$ is a representative 1-cocycle of a cohomology generator.

Let A_k denote the set of edges that define a cocycle in $\bar{G}_k$ (the boundary graph in level k). The **down projection** of A_k in $\bar{G}_{k-1}$ is $A_{k-1} = A^s_{k-1} \cup A^r_{k-1}$, where A^s_{k-1} denotes the set of surviving edges in $\bar{G}_{k-1}$ that correspond to A_k, and A^r_{k-1} is a subset of removal edges in $\bar{G}_{k-1}$. The following steps show how to obtain A^r_{k-1}:

1. Consider only the contraction kernels of G_{k-1} (the adjacency graph) whose vertices are labeled with label ℓ. The edges of each contraction kernel are oriented toward the respective root such that each edge has a unique starting vertex.

2. For each contraction kernel T. From the leaves of T to the root. Let e be an edge of T and v its starting point: label e with the number of edges that are in both A^s_{k-1} and the set of edges that are in the boundary of the face associated to v, plus the sum of the labels of the edges of T which are incident to e.

3. A removal edge of $\bar{G}_{k-1}$ is in A^r_{k-1} if the corresponding edge of G_{k-1} is labeled with an odd number.

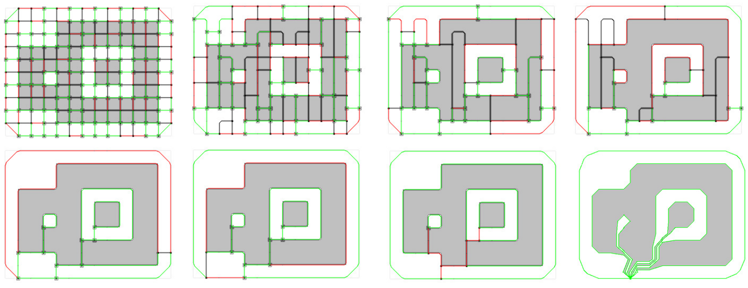

Figure 4: The boundary graphs of all the levels of an irregular graph pyramid (level 0 on the top-left) and the homology-generator level (on the bottom-right). Removed edges in black, contracted edges in red and surviving edges in green.

The proof of the correctness of the algorithm is done using the homomorphisms $\{\pi_p\}$. These maps are defined from the integral operators associated to the removal and contracted edges made in the boundary graph of level $k-1$ to obtain level k. An example of the down projection of a representative 1-cocycle is shown in Figure 5.

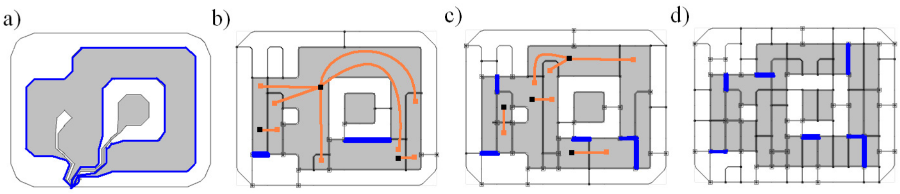

Figure 5: a) The boundary graph of the homology-generator level; in blue, the edges that define a representative 1-cocycle c. b) and c) The boundary graphs of levels 3 and 2 with the contraction kernels drawn in orange (the roots of the trees are drawn in black). d) In blue, the edges that represent the down projection of the representative 1-cocycle c in the boundary graph of level 2.

It is worth to be mentioned that the obtained generators by this method will be as good or bad as the choices of edge removal which have been made at every step.

5 Discussion

It is expected that the algorithm presented in Section 4 is also valid in 3D, using for example combinatorial [7, 1] or generalized map pyramids [4]. Lemma 1 has to be valid for the pyramid building process, and Lemma 2 has to be extended for the used operations (e.g. face contraction, face removal, etc.). The method can also be used to compute finer invariants such as the cohomology ring or cohomology operations in the same way as it is done in [3]. For dimensions higher than three, the homology and cohomology groups are no longer isomorphic and the extension of the method to these dimensions is no longer straight forward.

Having interpretations in graph theory of concepts related to cohomology opens the door for applying classical graph theory algorithms. For example, any set of foreground edges in the boundary graph $\bar{G}_i$ associated to a path in the adjacency graph G_i connecting a hole of the object with the (outside) background face is a representative 1-cocycle, and this can be computed efficiently. If additional constrains are added, like minimal length, the 1-cocycle is a good candidate for pattern recognition tasks as it is invariant to the scanning of the cells, the processing order, rotation, etc. The interpretation of cocycles with graph theory concepts, has to cover also higher dimensions (e.g. what are 1-cocycles of a 3D object? what would be a minimal 1-cocycle?).

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