Line Voronoi Diagrams Using Elliptical Distances

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Abstract. The paper introduces an Elliptical Line Voronoi diagram. In contrast to the classical approaches, it represents the line segment by its end points, and computes the distance from point to line segment using the Confocal Ellipse-based Distance. The proposed representation offers specific mathematical properties, prioritizes the sites of the greater length and corners with the obtuse angles without using an additional weighting scheme. The above characteristics are suitable for the practical applications such as skeletonization and shape smoothing.

Keywords: Confocal ellipses · Line Voronoi diagram
Hausdorff distance

1 Introduction

Various branches of computer science - for example, pattern recognition, computer graphics, computer-aided design - deal with the problems that are inherently geometrical. In particular, Voronoi diagram is a fundamental geometrical construct that is successfully used in a wide range of computer vision applications (e.g. motion planning, skeletonization, clustering, and object recognition) [1]. It reflects the proximity of the points in space to the given site set.

On one side, proximity depends on a selected distance function. Existing approaches in $\mathbb{R}^2$ explore the properties and application areas of particular metrics: L_1 [2], L_2 [3,4], L_p [5]. Chew et al. [6] present the Voronoi diagrams for the convex distance functions. Klein et al. [7] introduced a concept of defining the properties of the Voronoi diagram for the classes of metrics, rather than analyzing each metric separately. A group of approaches proposes the site-specific weights, e.g. skew distance [8], power distance [9], crystal growth [10], and convex polygon-offset distance function [11]. This paper presents a new type of a Line

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Voronoi diagram that uses Confocal Ellipse-based Distance (CED) [12] as a metric of proximity. In contrast to Hausdorff Distance (HD), CED (1) defines the line segment by its two end points, (2) represents the propagation of the distance values from the line segment to the points in $\mathbb{R}^2$ as confocal ellipses. The proposed geometrical construct reconsiders the classical Euclidean distance-based space tessellation, and introduces hyperbolic and elliptical cells, that have surprising mathematical properties. Structure is added to a set of points by putting the subsets of points in relation. The simplest relation that every structure should have is a binary relation relating two points. That is why a new metric relating points with pairs of points is extremely relevant for the community.

On the other side, proximity depends on the type of objects in the site set. Polygonal approximations of objects are commonly agreed to be used in a majority of geometric scenarios [13]. Therefore, in this paper the site set contains points and/or line segments.

The remainder of the paper is organized as follows. Section 2 presents the Elliptical Line Voronoi diagram (ELVD), provides an analysis of the proximity as defined by CED and HD, and introduces the Hausdorff ellipses. Section 3 shows the properties of ELVD with regard to the type of objects in the site set. Section 4 discusses the advantages of applying the ELVD to skeletonization and contour smoothing. Finally, the paper is concluded in Sect. 5.

2 Elliptical Line Voronoi Diagram (ELVD)

A Voronoi diagram partitions the Euclidean plane into *Voronoi cells* that are connected regions, where each point of the plane is closer to one of the given *sites* inside the cell. In the classical case the sites are a finite set of points and the metric used is the Euclidean distance.

In our contribution we extend the original definition by (1) considering a site to be a straight line segment, (2) measuring the proximity of a point to the site using the parameters of a unique ellipse that passes through this point and takes the two end points of the line segment as its focal points. We call the resultant geometrical construct Elliptical Line Voronoi diagram, or in short ELVD.

As opposed to Euclidean distance in Voronoi diagram, proximity in the ELVD is defined with respect to the Confocal Ellipse-based Distance. Similarly to the Blum's medial axis [14], ELVD can be extracted from the Confocal Elliptical Field (CEF) [12] as a set of points which have identical distance value for at least two sites.

2.1 Confocal Ellipse-Based Distance (CED)

Let $\delta(M, N) = \sqrt{(M - N)^2}, M, N \in \Re^2$, be the Euclidean distance between the points M and N.

Definition 1. *The ellipse, $E(F_1, F_2; a)$*[1] *is the locus of points on a plane, for which the sum of the distances to two given points F_1 and F_2 (called focal points) is constant:*

$$\delta(M, F_1) + \delta(M, F_2) = 2a, \tag{1}$$

where parameter a is the length of the semi-major axis of the ellipse.

Ellipses that have the same focal points F_1 and F_2 are called confocal ellipses. Given two focal points F_1 and F_2, a family of confocal ellipses covers the whole plane. Each ellipse in this family is defined as $E(a) = \{P \in \Re^2 |\ \delta(P, F_1) + \delta(P, F_2) = 2a\}$, $a \geq f$. Here $f = \frac{\delta(F_1, F_2)}{2}$ denotes half the distance between the two focal points F_1 and F_2.

Definition 2. *Let us consider two confocal ellipses $E(a_1)$ and $E(a_2)$ generated by focal points $F_1, F_2 \in \Re^2$, where $a_1,\ a_2 \geq f$. The Confocal Ellipse-based Distance (CED) between $E(a_1)$ and $E(a_2)$, $e : \Re^2 \times \Re^2 \to \Re$, is determined as the absolute difference between the lengths of their major axes:*

$$e(E(a_1), E(a_2)) = 2|a_1 - a_2| \tag{2}$$

CED is a metric and $E(a_1) \subset E(a_2)$, if $a_1 < a_2$.

2.2 Confocal Elliptical Field (CEF)

Consider a set of sites that contains the pairs of points: $S = \{(F_1, F_2), (F_3, F_4), ..., (F_{N-1}, F_N)\}$. A site $s = (F_i, F_{i+1}), i \in [1, ..., N-1]$ generates a family of confocal ellipses with F_i and F_{i+1} taken as the focal points. The distance from the point $P \in \Re^2$ to the site s, is defined with respect to CED as:

$$d(P, s) = e(E(a_P), E(a_0)) \tag{3}$$

where $E(a_P)$ corresponds to the unique ellipse with focal points F_i and F_{i+1} that contains P; $E(a_0)$ corresponds to the ellipse with the same foci F_i and F_{i+1}, whose eccentricity equals 1. In other words, this distance is defined as: $d(P, s) = \delta(P, F_i) + \delta(P, F_{i+1}) - \delta(F_i, F_{i+1}) = 2(a - f)$.

Definition 3. *Confocal Elliptical Field (CEF) is an operator that assigns to each point $P \in \Re^2$ its distance to the closest site from S:*

$$CEF = d(P, S) = inf\{d(P, s) \mid s \in S\} \tag{4}$$

Definition 4. *Separating curve is a set of points in CEF that have an identical value as generated from multiple (more than one) distinct sites.*

For the given set of sites that contain points and line segments, separating curves define the ELVD.

[1] If for several ellipses the focal points are the same, we denote it as $E(a)$.

2.3 Relation Between CED and Hausdorff Distance

As opposed to CEF, in classical Line Voronoi diagram, the line segment is a set of all points that form it. Therefore, for each point in space the proximity to the line segment can be defined with respect to the Hausdorff Distance.

Definition 5. *The Hausdorff Distance (HD) between a point P and a set of points T is defined as the minimum distance of P to any point in T. Usually the distance is considered to be Euclidean:*

$$HD = d_H(P, T) = inf\{\delta(P, t) \mid t \in T\} \tag{5}$$

By introducing a scaling factor of $\frac{1}{2}$ for the CED we obtain the same distance field for HD and CED, in case the two focal points coincide. Another property is that the λ-isoline of the CED $\{P|d(P, s) = \lambda\}$ encloses the r-isoline of HD $\{P|d_H(P, T) = r\}$, with s being a site containing the two foci F_1 and F_2, T is a set of points that form the line segment $\overline{F_1F_2}$. Figure 1a shows multiple isolines for HD and CED that have the same λ and r. Note that both, HD and CED, have zero distance values along the line segment $\overline{F_1F_2}$.

We can derive a value λ for any given r so that the CED λ-isoline is enclosed by the HD r-isoline (see Fig. 1b). To find λ we are looking for the value where the minor ellipse radius b equals r. In an ellipse $b^2 = a^2 - f^2$, that in this case can be reformulated to $r^2 = a^2 - f^2$, solving for a:

$$\lambda = 2a - f = 2\sqrt{r^2 + f^2} - f. \tag{6}$$

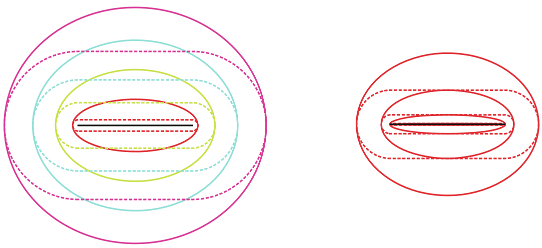
By similar reasoning we can also derive r for a given λ that will ensure the r-isoline of the HD is enclosed by the CED λ-isoline:

$$r = \sqrt{2f\lambda + \lambda^2}. \tag{7}$$

We can construct ellipses around a line segment by starting with a distance $\lambda_0 = 1$ and increasing according to the sequence:

$$\lambda_{n+1} = \sqrt{2f\lambda_n + \lambda_n^2} \tag{8}$$

We name these isolines *Hausdorff Ellipses* of a line segment.



(a) $\lambda = r$ (b) $\lambda = 2\sqrt{r^2 + f^2} - f$

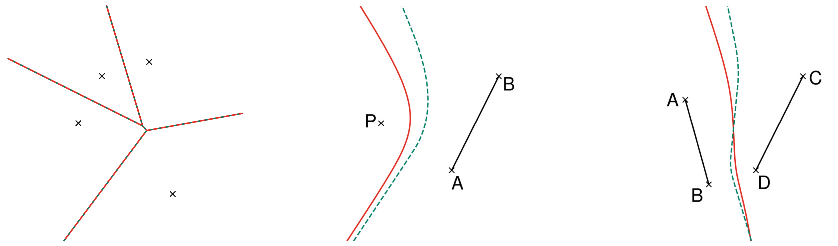
Fig. 1. Comparison of HD (dashed) and CED (solid) isolines

3 Properties of ELVD

The proximity depends not only on the type of metric used, but also on the type of object in the site set. In this paper site is considered to be a point or a line segment. According to the Definition 3 of CEF, the distance field of a point contains concentric circles, and of a line segment - confocal ellipses. Thus, the separating curve varies according to the different combinations of the site types.

3.1 Point and Point

In terms of CED, the site that represents a point contains identical foci. The resultant distance field of each site is formed by concentric circles. The separating curves are the perpendicular bisectors, and the ELVD is identical to the Voronoi diagram with Euclidean distance (Fig. 2a).



(a) Point-Point (b) Point-Line (c) Line-Line

Fig. 2. Comparison of ELVD (solid red) and Voronoi diagram (dashed green). (Color figure online)

3.2 Point and Line

Consider the site set that contains point P and line segment (A, B). The receptive field of the point P depends on the position of the line segment, and ELVD is represented by a higher-order curve (Fig. 2b).

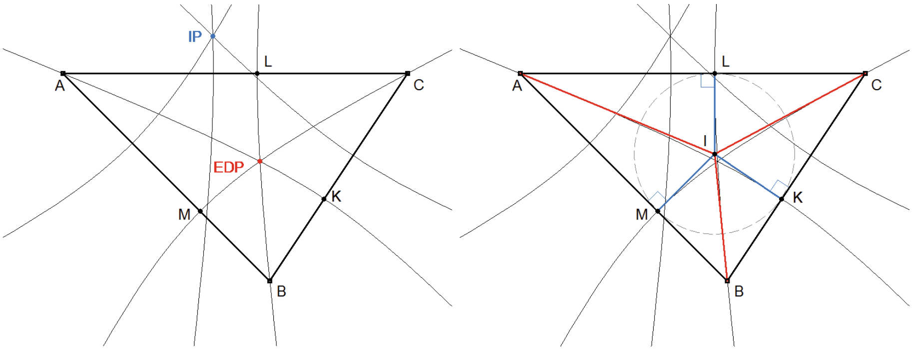
3.3 Line and Line

For the site set that contains two line segments (A, B) and (C, D), the ELVD is represented by a high-order curve of a different nature than for the Point-Line case (see Fig. 2c). The steepness and the shape of the curve depends on the length of the line segments, and their mutual arrangement (parallel, intersecting, non-intersecting). The mutual arrangement does not consider (A, B) and (C, D) to be connected as a polygon, i.e. $B \neq C$. This case is covered in Sect. 3.5.

3.4 Triangle

The simplest closed polygonal shape - a triangle - can be represented by:

– *three points corresponding to its vertices*
  In the classical Voronoi diagram on the point set, the separation curves of the (Delaunay-) triangle are the perpendicular bisectors of its edges, they intersect at the center of the circumscribed circle.
– *by a set of N points, that form the contour of the triangle*
  In the extension of the classical Line Voronoi diagram on the line set using the Euclidean distance, the separating curves of the triangle are its angular bisectors which intersect at the center of the incircle.
– *by three line segments corresponding to the edges of the triangle*
  For the ELVD the separating curve between the two line segments that share one endpoint is a hyperbolic branch [12]. Therefore, the separation curves in the triangle are three hyperbolic branches, each passing through one vertex of the triangle, i.e. A, B or C, and intersecting the sides at the points K, L, M respectively (Fig. 3a).



(a) Hyperbolic branches of the ELVD intersect at the Equal Detour Point (EDP) and Isoperimetric Point (IP).

(b) The tangents on the hyperbola in the intersection points A, B, C and K, L, M intersect at the incircle center (I).

Fig. 3. Properties of the Equal Detour Point, Isoperimetric Point and incenter.

The separating curves of the triangle as obtained from ELVD have the following geometric properties:

1. The separating curves intersect at a common point, known in the literature as the *Equal Detour Point (EDP)* [15] (see Fig. 3a).
2. The complementary branches of the hyperbolas intersect at a common point, known as the *Isoperimetric Point (IP)* [15] (Fig. 3a).
3. The six tangents of the hyperbolas at the six points A, B, C, and K, L, M intersect all at the center of the incircle I (Fig. 3b).

4. The intersection EDP of the three hyperbolas is located inside the triangle formed by the shortest side of the triangle and I (Fig. 3b).
5. The tangents at the triangle's corners A, B, C are the angular bisectors of the two adjacent sides respectively (Fig. 3b).
6. The three tangents at K, L, M form a right angle while intersecting the edges of the triangle (Fig. 3b).
7. The hyperbola chords AK, BL and IM intersect at the *Gergonne point (G)* [15] (Fig. 4).
8. The EDP distance value of the CEF equals the radius of the inner Soddy circle.
   Let $P \in \mathbb{R}^2$ be an EDP, and K, L, M - be the points of intersection between separating curves and the edges of the triangle $\triangle ABC$. Consider the following distances: (1) $r_P = CEF(P)$ - distance value at P in the confocal elliptical field; (2) $r_A = \delta(A, M) = \delta(A, L)$; (3) $r_B = \delta(B, M) = \delta(B, K)$; (4) $r_C = \delta(C, L) = \delta(C, K)$. The circle with the center at P and radius r_P is an inner Soddy circle [16], thus, it is tangent to the circles with the centers at A, B, C and radii r_A, r_B, r_C correspondingly. This property is valid not only for the EDP, but for all points of the separation hyperbola branches that lie on the curves PM, PK, and PL. In addition, according to the Soddy theorem, the following equation holds true:

$$\left(\frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} + \frac{1}{r_P}\right)^2 = 2\left(\frac{1}{r_A^2} + \frac{1}{r_B^2} + \frac{1}{r_C^2} + \frac{1}{r_P^2}\right) \tag{9}$$

   In case of a regular triangle, radii r_A, r_B, r_C are identical. Otherwise, their values vary depending on the angle at the corresponding vertex, and length of the edges that contain this vertex. The ELVD implicitly encodes the weighting factors, as compared to the classical Voronoi diagram.

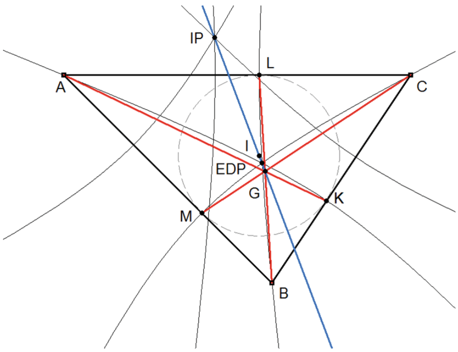


Fig. 4. The incenter (I), Gergonne point (G), Isoperimetric Point (IP) and Equal Detour Point (EDP) are collinear.

3.5 Polygon

Consider a site set that defines an open polygon $S = \{(F_1, F_2), \ldots, (F_{N-1}, F_N)\}$, $N \in \mathbb{R}$. For any $s_i = (F_i, F_{i+1})$, $F_i \neq F_{i+1}$, $s_i \in S$, $i \in [1, N-1]$.

If the sites are consecutive, i.e. have a common point F_i, the separating curve is a branch of a hyperbola that passes through F_i, $i \in [1, N]$ [12]. If the sites are non-consecutive, but their receptive fields overlap (e.g. the sites cross each other), then the separating curve is defined as in *Line and Line* case.

Let P be the point of intersection of two separating curves H_{F_i} and $H_{F_{i+1}}$, that pass through F_i and F_{i+1} correspondingly. For the triangle $\triangle F_i P F_{i+1}$ the separation hyperbola branch that passes through P and intersects (F_i, F_{i+1}) at the point M defines the following distances: $r_{F_i} = \delta(F_i, M), r_{F_{i+1}} = \delta(F_{i+1}, M)$. The circle with the center at P and radius r_P is tangent to the circles with centers at F_i, F_{i+1} and radii $r_{F_i}, r_{F_{i+1}}$ respectively. This property holds true for all points on the separating curve between P and M.

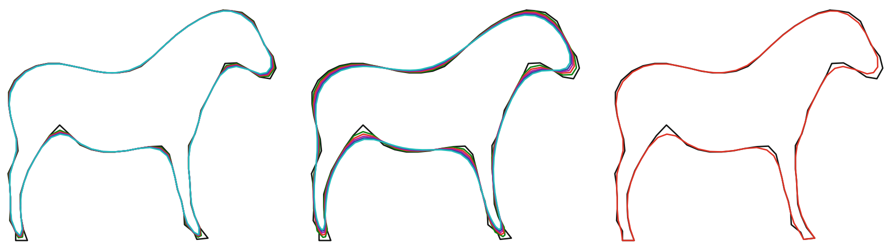
4 Applications

In this section we discuss the properties of ELVD that are valuable for the practical problems on an example of contour smoothing and skeletonization.

4.1 Contour Smoothing

By considering three successive points P_{i-1}, P_i and P_{i+1} on a contour as a triangle $\varDelta_i$ we can smooth the contour by replacing the middle point P_i with the EDP of the triangle $\varDelta_i$. Conventional average smoothing is related to the centroid of the triangle $\varDelta_i$. This smoothing procedure can be iteratively repeated. Figure 5 shows a comparison between EDP-based smoothing and Mean-based smoothing, i.e. averaging over three successive contour points. Note that EDP-based smoothing does not affect low frequencies as much as high frequencies.

Let us denote the angles in the triangle $\varDelta_i$ as α, β, γ. The angles formed by the vertices of the triangle and the incenter are $\frac{\pi+\alpha}{2}, \frac{\pi+\beta}{2}, \frac{\pi+\gamma}{2}$. This means



(a) EDP-based smoothing (b) Mean-based smoothing (c) Preserved sharp corners

Fig. 5. Contour smoothing achieved by five iterations.

that the sharp angle ($<\frac{\pi}{2}$) will be replaced by the obtuse angle after smoothing. The shortest side has the smallest opposite angle and an angle of more than $\frac{\pi}{2}$ is always the largest in a triangle. Hence: (1) the shortest side before smoothing becomes the longest, (2) the smoothing slows down with more iterations. According to the ELVD Properties 4 and 8, in case of a triangle, the same holds true for the EDP. The difference is that the incenter is equidistant from the corner sides, whereas EDP is closer to the shorter edge and obtuser angle than the incenter. This property is important in case of the outliers - the contour is smoothed with the less number of iterations.

Additionally we can preserve selected sharp corners by including the same point twice in the contour. Figure 5c gives an example of preserved sharp corners in the hooves of the horse.

4.2 Skeletonization

The ELVD can be successfully applied to create a skeleton of the shape [12], where the weighting is implicitly encoded in the length of the site (see Fig. 6). As compared to the classical Voronoi diagram-based skeletonization, the sites contain pairs of vertices. The skeletal points are not equidistant from the opposite sides of the shape - they are shifted towards the sites that represent the shorter edges. As a result, the longer edges have a greater receptive field.

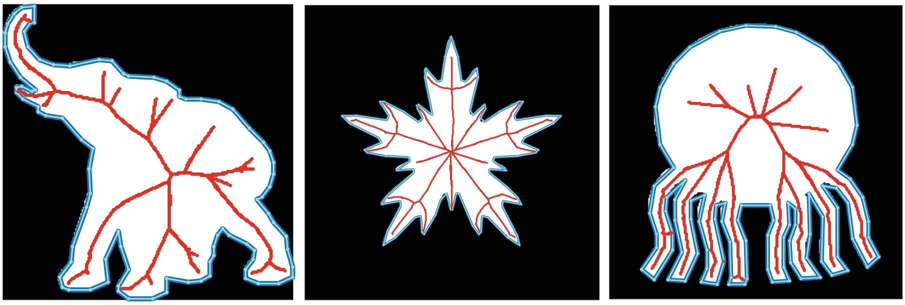


Fig. 6. Examples of the ELVD-based skeletons (red). The polygonal approximation of the shape (cyan) contains 90 vertices in each case. (Color figure online)

5 Conclusion and Outlook

This paper presents a novel approach to the line Voronoi diagram by considering the distance from the point to the line segment by CED. The discussion of the ELVD proximity (from the point of metric and types of objects in the site set) shows that the classical Voronoi diagram is a special case of ELVD. The proposed approach has also the practical value: (1) skeletonization algorithm enables prioritization of the longer edges without extra weighting schema, (2) smoothing of the shape enables a closer approximation of the contour and preservation of the sharp corners. The ongoing research considers ELVD properties regarding the weighting factors and the semantic interpretation of the corresponding geometrical construct.

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