Skeletons in the Framework of Graph Pyramids ⋆

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Abstract. Graph pyramids allow to combine pruning of skeletons with a concept known from the representation of line images, i.e. generalization of paths without branchings by single edges. Pruning will enable further generalization of paths and the latter speeds up the former. Within the unified framework of graph pyramids a new hierarchical representation of shape is proposed that comprises the skeleton pyramid, as proposed by Ogniewicz. In particular, the skeleton pyramid can be computed in parallel from any distance map.

1 Introduction

A major goal of skeletonization consists in bridging the gap between low level raster-oriented shape analysis and a semantic object description [Ogn94]. In order to create a basis for the semantic description, the medial axis [Blu62,Ser82] is often transformed into a plane graph [Ogn94]. This task has been solved using the Voronoi diagram defined by the boundary points of a shape [Ogn93,Ogn94,OK95] or by the use of special metrics on derived grids [Ber84]. In this paper we will propose a method that is not confined to a special metric (distance map) on a special grid nor on a special irregular structure like the Voronoi diagram.

The new method starts with a regular or irregular *neighborhood graph*. The neighborhood graph reflects the arrangement of the sample points in the plane. The vertices of the neighborhood graph represent the sample points and the distances from the sample points to the boundary of the shape are stored in the vertex attributes. The edges of the neighborhood graph represent the neighborhood relations of the sample points. All illustrations in this paper refer to the regular neighborhood graph, in which the sample points represent pixel centers and the edges indicate the 4-connectivity of the pixels (Fig. 1a). The vertex attributes reflect the Euclidean distance map (EDM) on a 4-connected set S of

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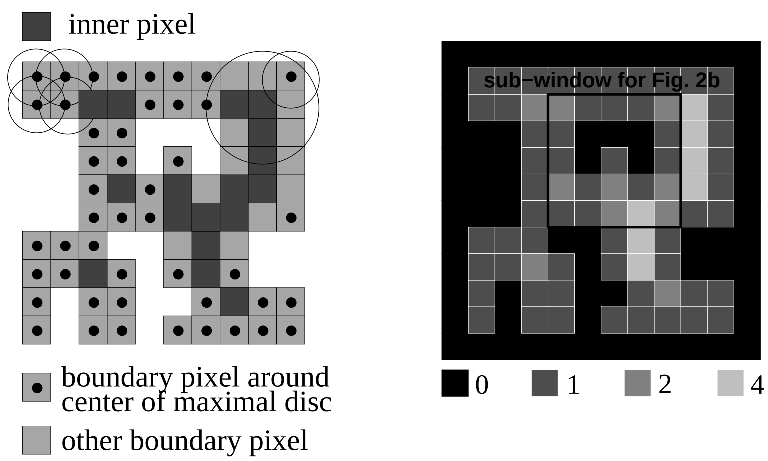
pixels: each pixel p of S is equipped with the Euclidean distance between the center of p and the *closest pixel center outside of* S [SL98] (Fig. 1b).

The dual of the neighborhood graph is referred to as *crack graph*. The edges of the crack graph describe the borders of the pixels. Each edge e of the neighborhood graph perpendicularly intersects exactly one edge $\overline{e}$ of the crack graph. The edge e is called the *dual* of $\overline{e}$ and vice versa. Each vertex of the crack graph stands for a pixel corner (Fig. 2b).

Dual graph contraction (DGC) [Kro95] is used to successively generalize the neighborhood graph by the removal and the contraction of edges. One level of the resulting *graph pyramid* will be called *skeleton graph* (Fig. 5a). This term is justified by the fact that all centers of maximal disks (with respect to the distance map) are represented by vertices of the skeleton graph. Furthermore, the skeleton graph is always connected.

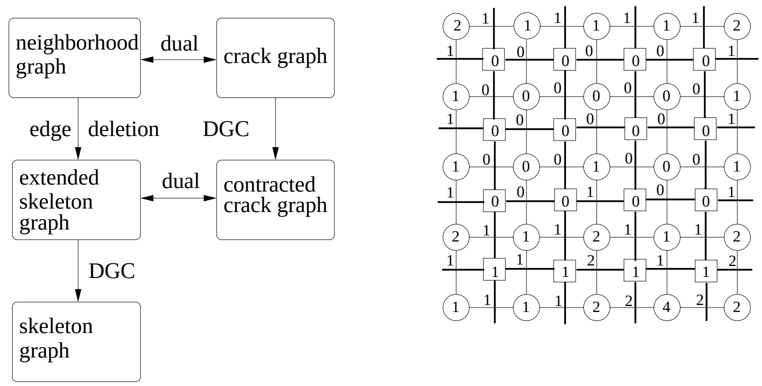


(a) (b)

Fig. 1. (a) 4-connected pixel set. (b) Squared distances of (a).

This paper is organized as follows: Section 2 is devoted to the initialization of the attributes in the neighborhood graph and in the crack graph. In Section 3 the crack graph is contracted. Regarding the neighborhood graph this amounts to the deletion of edges that are dual to the ones contracted in the crack graph. The reduced neighborhood graph is called *extended skeleton graph* (since it contains the skeleton graph). In Section 4 the extended skeleton graph is contracted to the skeleton graph. An overview of the different graphs and their relations is given in Fig.2a.

Like the skeleton, the skeleton graph is not robust. In Section 5 we propose a pruning and generalization method for the skeleton graph. It is based on DGC and yields the new shape representation by means of a graph pyramid. The



(a) (b)

Fig. 2. (a) Overview of the graphs and their relations. (b) Neighborhood graph (○) and crack graph (□) restricted to the sub-window in Fig. 1b. The numbers indicate the attribute values of the vertices and the edges.

pyramid proposed in [OK95] can be obtained from the new representation by threshold operations. We conclude in Section 6.

2 Initialization of the Neighborhood Graph and the Crack Graph

The neighborhood graph may be interpreted as digital elevation model (DEM), if the vertex attributes, i.e. the distances of the corresponding sampling points to the border of the shape, are interpreted as altitudes. Intuitively, the plan for the construction of the skeleton graph is to reduce the neighborhood graph such that the remaining edges describe the connections of the summits in the DEM via the crest lines of the DEM. In contrast to [KD94,NGC92] our concept is a dual one: the neighborhood relations of the basins are described by the dual of the skeleton graph. In the next two sections it will turn out that the reduction of the skeleton graph depends only on the order of the values from the distance transform. Hence, we may use *squared* distances and thus avoid non-integer numbers. The idea for the reduction of the neighborhood graph is to remove edges that do not belong to ridges - thus forming the basins represented by the dual graph. The following initialization (Fig. 2b) will allow to control this process. The first part refers to the neighborhood graph, the second to the crack graph (Fig. 2b):

- Let $dist^2(v)$ denote the squared distance of the pixel that corresponds to vertex v. The attribute value of v is set to $dist^2(v)$. The attribute value of edge $e = (u, v)$ is set to the minimum of $dist^2(u)$ and $dist^2(v)$.

– The attribute value of edge $\overline{e}$ is set to the attribute value of edge e, where e denotes the edge in the neighborhood graph that is dual to $\overline{e}$. The attribute value of vertex $\overline{v}$ is set to the minimum of the attribute values of all edges incident to $\overline{v}$.

3 Contracting the crack graph

Recall, that the contraction of an edge in the crack graph is associated with the removal of the corresponding dual edge in the neighborhood graph [BK99a]. The neighborhood graph can never become disconnected: The removal of an edge e would disrupt the neighborhood graph, only if the corresponding dual edge $\overline{e}$ in the crack graph was a self-loop. DGC, however, forbids the contraction of self-loops.

In order to get an intuitive understanding of the duality between contraction and deletion, we focus on the embedding of graphs on the plane (only planar graphs can be embedded on the plane) [TS92]. An embedding of the neighborhood graph on the plane divides the plane into *regions* (Fig.3). Note that the

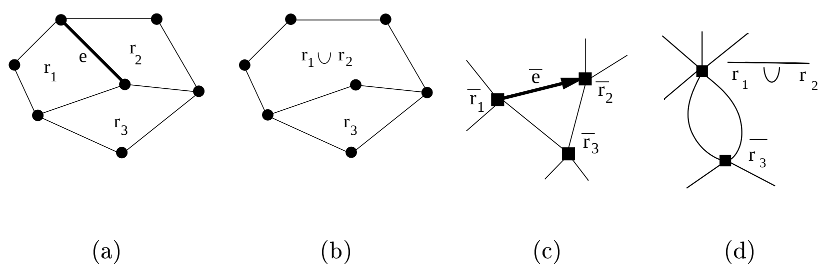


Fig. 3. Duality of edge deletion in a plane graph (a)→(b) and edge contraction in the dual of the plane graph (c)→(d). The regions $r_1, \ldots, r_4$ in (a) are represented by the vertices $\overline{r_1}, \ldots, \overline{r_4}$ in (c).

removal of an edge in the neighborhood graph is equivalent to the fusion of the regions on both sides of the edge. In terms of watersheds [MR98] it is intuitive to fuse the regions of the neighborhood graph until each of the resulting regions corresponds to exactly one basin of the landscape. **Two neighboring regions may be fused, if there is no separating ridge between the regions**. Due to the initialization of the attribute values in the crack graph, we may formulate a criterion for the fusion of two regions as follows [GEK99]: Let r_1 and r_2 denote two regions of the neighborhood graph and let $\overline{r_1}$ and $\overline{r_2}$ denote the corresponding vertices in the crack graph. The regions r_1 and r_2 ($r_1 \neq r_2$) may be fused, if there exists an edge $\overline{e}$ between $\overline{r_1}$ and $\overline{r_2}$, whose attribute value equals the attribute value of $\overline{r_1}$ or the attribute value of $\overline{r_2}$. Assume that the attribute value of $\overline{r_2}$ is smaller or equal to the attribute value of $\overline{r_2}$. Then the fusion of r_1 and

r_2 is achieved by the contraction of $\overline{r_1}$ into $\overline{r_2}$ (Fig. 3c). Thus, during the whole contraction process the attribute values of a vertex in the crack graph indicates the altitude of the deepest point in region represented by the vertex.

Multiple fusions can be done by iterating the following *parallel* steps:

1. For each edge of the crack graph that meets the above criterion for contraction, mark the end vertex with the minimal attribute value. In case of equality choose one of the end vertices by a random process.
2. Form a maximal independent set (MIS) of the marked vertices as explained in [Mee89]. The MIS is a maximal subset of the marked vertices, no two elements of which are connected by an edge.
3. Contract all edges that are incident to a vertex v of the MIS and that meet the above criterion for contraction (v being the end vertex with the minimal attribute).

The iteration stops, when none of the edges in the crack graph meets the above criterion. The resulting graph is called *extended skeleton graph* (Fig. 5a). It is connected and it still contains all vertices of the neighborhood graph.

4 Contracting the neighborhood graph

In this section the extended skeleton graph is further reduced to the so called *skeleton graph*. The skeleton graph

- still must contain all vertices which represent maximal discs and
- still must be connected.

We focus on edges $e = (u, v)$ such that v has degree 1 in the extended skeleton graph. The idea is to contract v into u, if we can tell by a local criterion that v does not represent a center of a maximal disc. All edges that have a degree-one end vertex to fulfill this criterion may then be contracted in parallel.

Consider an edge $e = (u, v)$ such that v is the end vertex with degree one. Using the notation of Section 2, i.e. $dist(v)$ [$dist^2(v)$] for the [squared] distance of a vertex v, we formulate the following criterion: If

$$dist(u) - dist(v) = 1, \tag{1}$$

the vertex v does not represent a center of a maximal disc and v may be contracted into u [San94].

The distances $dist(u)$ and $dist(v)$ in condition(1) are integers [1]. This follows from equation

$$dist^2(u) = (dist(v) + 1)^2 = dist^2(v) + 2dist(v) + 1 \tag{2}$$

and the fact that the squared distances are integers.

[1] Thus condition(1) may be checked using only the squared distances and a look-up table.

In case of grids other than the square grid or in case of irregular samplings, Equation 1 generalizes to

$$dist(u) - dist(v) = \| u - v \|_2, \tag{3}$$

where $\| \cdot \|_2$ denotes the Euclidean length of $u - v$. In terms of [Ser82], u is *upstream* of v. Repeated contraction of edges in the extended skeleton graph yields the skeleton graph (Fig. 5a).

5 A New Hierarchical Representation for Shapes

The new hierarchy is build on top of the skeleton graph. Besides pruning we also apply generalization of paths between branchings by single edges, as proposed in [BK99b].

In order to asses the prominence of an edge e in the skeleton of a shape S, Ogniewicz [OK95]

1. defines a measure m for boundary parts b of S , i.e. the length of b,
2. for each edge e determines the boundary part b_e of S associated with e (this is formulated within the concept of Voronoi Skeletons),
3. sets the *prominence* of e to $m(b_e)$.

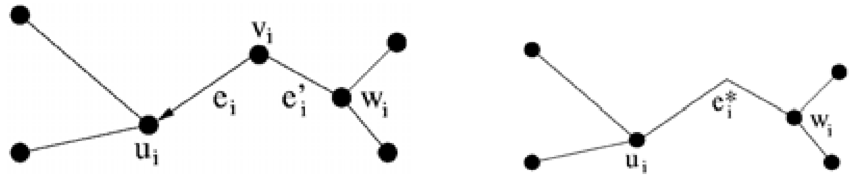
In our approach we also measure boundary parts by their lengths. However, we associate the boundary parts with vertices and thus define prominence measures for vertices. The initial prominence measure $prom(v)$ of v indicates the number of boundary vertices (vertices representing boundary pixels) contracted into v including v itself, if v is a boundary vertex. (Fig. 5a). This can already be accomplished during the contraction of the neighborhood graph (Section 4) by

1. setting $prom(v)$ to 1, if v is a boundary vertex, 0 otherwise (before the contraction),
2. incrementing the prominence measure of w by the prominence measure of v, if v is contracted into w.

Prominence measures will only be calculated for vertices that do not belong to a cycle of the skeleton graph. In the following, the calculation of the prominence measures from the initial prominence measures is combined with the calculation of the skeleton pyramid.

Let the degree of a vertex v in a graph be written as $deg(v)$ and let P denote a maximal path without branchings in the skeleton graph, i.e. $P = (v_1, v_2, \ldots, v_n)$ such that

- v_i is connected to v_{i+1} by an edge e_i for all $1 \leq i < n$ and
- $deg(v_i) = 2$ for all $1 < i < n$ and
- $deg(v_1) \neq 2$, $deg(v_n) \neq 2$.



(a) Before concatenation. (b) After concatenation.

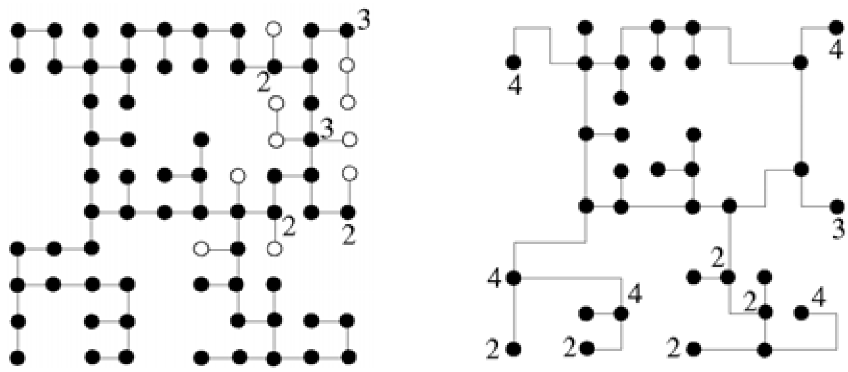
Fig. 4. Concatenation of edges.

Let $e_i = (u_i, v_i)$ be an edge of P with $u_i \neq v_i$ and $deg(v_i) = 2$ (Fig. 4a). Since $deg(v_i) = 2$, there is a unique edge $e_i' \neq e_i$ in P with $e_i' = (v_i, w_i)$, $w_i \neq v_i$. We assume that e_i does not belong to a cycle, i.e. $w_i \neq u_i$. If $deg(w_i) \neq 1$, we allow that v_i may be contracted into u_i. The contraction of e_i can be described by the replacement of the two edges e_i and e_i' by a new edge e_i^* (Fig. 4b).

The prominence measure of u_i is updated by

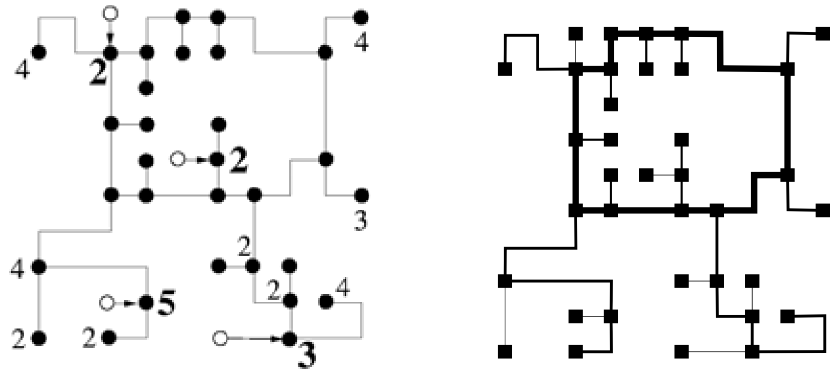
$$prom(u_i) := prom(u_i) + prom(v_i). \tag{4}$$

This contraction process is referred to as *concatenation*. Due to the requirement $deg(w_i) \neq 1$ the prominence measures are successively collected at the vertices with degree 1. The result of concatenation on the skeleton graph in Fig. 5a is depicted in Fig. 5b.



(a) (b)

Fig. 5. (a) Extended skeleton graph. The vertices of the skeleton graph are given by the filled circles. The numbers indicate the initial prominence measures > 1. (b) After concatenation of the skeleton graph in Fig. 5a. The numbers indicate the prominence measures > 1.



(a) (b)

Fig. 6. (a) Contraction of vertices with degree 1. The numbers indicate the prominence measures > 1. (b) Ranks: bold 1, medium 2, thin 3.

After concatenation there are no vertices with degree 2. We focus on the set of *junctions with dead ends*, i.e. the set U of all vertices u with

- $deg(u) > 2$ and
- there exists an edge e and a vertex v with $e = (u, v)$ and $deg(v) = 1$.

The set of all edges that connect a vertex $u \in U$ with a vertex v of degree 1 is denoted by $Ends(u)$. Note, that for each edge e there is at most one $u \in U$ with $e \in Ends(u)$. For each $u \in U$ let $e_{min}(u)$ denote an edge in $Ends(u)$, whose end vertex with degree 1 has a minimal prominence measure.

The prominence measures of the vertices with degree 1 induce an order in $Ends(u)$, $e_{min}(u)$ being the least element. In case of $\mid Ends(u) \mid > 1$ we allow $e_{min}(u)$ to be contracted. Analogous to concatenation the prominence measure of u is updated to $prom(u) := prom(u) + prom(v)$.

In Fig. 6 the contraction is indicated by arrows: a white vertex at the source of an arrow is contracted into the black vertex at the head of the arrow. The new prominence measures are emphasized.

For each set $Ends(u)$ the operation of contraction followed by updating the prominence measure takes constant time. These operations can be performed in parallel, since the sets $Ends(u)$, $u \in U$ are disjoint. *Generalization* of the skeleton graph consists of iterating concatenation followed by contraction (both include updating).

In [OK95] a hierarchy of skeleton branches is established by a skeleton traversal algorithm. The traversal starts at the most prominent edge and follows the two least steep descents (starting at each end vertex of the most prominent edge). The highest rank skeleton branch consists of all edges that have been traversed by this procedure. Skeleton branches of second highest rank originate from the highest rank branch and are also least steep descent (ignoring the edges of the highest rank branch). Skeleton branches of third highest rank and so on are defined analogously. The edges of the skeleton are labelled according to the rank of the skeleton branch they belong to.

Analogous ranks can be determined by imposing a restriction on the generalization described above: for each vertex $u \in U$ only one edge in $Ends(u)$ may be contracted. This is achieved by initializing all vertices as *vacant*. Once an edge from $Ends(u)$ was contracted, u is marked as *occupied*. The generalization up to the state, in which no further generalization can be done is summarized as *first step*. Thereafter, all vertices are marked as *vacant* again. The second step is finished, when occupation again forbids further generalization and so on. Thus, each edge of the concatenated skeleton graph that does not belong to a cycle is contracted. If n denotes the number of the last step, the *rank* of an edge contracted in step k, $(1 \leq k \leq n)$ is set to $2 + n - k$. Edges that belong to at least one cycle of the extended skeleton graph receive rank 1.

The set of edges with rank smaller or equal to k, $(1 \leq k \leq n + 1)$ always forms a connected graph. As in [OK95] these graphs can be derived by a simple threshold operation on the concatenated skeleton graph according to the ranks of the edges. The ranks of the extended skeleton graph in Fig. 5b are shown in Fig. 6c.

6 Conclusion

In this paper we have introduced a graph based hierarchical representation of shapes that comprises the skeleton pyramid as proposed by Ogniewicz. The new representation relies on the concept of graph pyramids by dual graph contraction. It allows to represent paths without branchings by single edges. This additional hierarchical feature is suggested for the hierarchical matching of shapes by means of their skeletons.

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