Technical Report

Dept. for Pattern Recognition and Image Processing
Institute for Automation
Technical University of Vienna
Treitlstr. 3/1832
A-1040 Vienna AUSTRIA
Phone: +43 (1) 58801-8161
Fax: +43 (1) 569697
E-mail: krw@prip.tuwien.ac.at

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Building Irregular Pyramids by Dual Graph Contraction[1]

Walter G. Kropatsch

Abstract

Many image analysis tasks lead to or make use of graph structures that are related through the analysis process with the planar layout of a digital image. This paper presents a theory that allows to build different types of hierarchies on top of such image graphs. The theory is based on the properties of a pair of dual image graphs that the reduction process should preserve, e.g. the structure of a particular input graph. The reduction process is controlled by decimation parameters, i.e. a selected subset of vertices, called survivors, and a selected subset of the graph's edges, the parent-child connections. It is formally shown that two phases of contractions transform a dual image graph to a dual image graph built by the surviving vertices. Phase one operates on the original (neighborhood) graph and eliminates all non-surviving vertices. Phase two operates on the dual (face) graph and eliminates all degenerated faces that have been created in phase one. The resulting graph preserves the structure of the survivors, it is minimal and unique with respect to the selected decimation parameters. The result is compared with two modified specifications, the one already in use for building stochastic and adaptive irregular pyramids.

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1 Introduction

The need for hierarchies in image analysis has been expressed by many scientists, e.g. recently by Nagy [17]. Multiresolution pyramids are already widely used in image analysis [18, 4, 23]. Hierarchies are motivated both by biological plausibility [22] and by computational efficiency [7].

Adjacency plays an important role in image analysis, too. Starting with the definition of neighboring pixels in low level processes up to adjacencies defined between regions resulting from segmentation processes, graphs can be used to represent these adjacency concepts. Although regular neighborhood structures dominate the lower levels of image processing and other data structures like arrays may be more efficient, at later processing stages regularity cannot be imposed.

Irregular pyramids combine graph structures with hierarchies. Similar to regular pyramids, we distinguish ordered levels of decreasing sizes in an irregular pyramid. Each level is a graph describing the image. Adjacent levels in decimation pyramids are related by the fact that the vertex set of the reduced level is a subset of the vertices in the level below. The methods for building irregular pyramids differ in several aspects:

1. in the way they select the survivors;

2. in the way they derive the neighborhood relations of the reduced level.

The first aspect may heavily depend on the kind of application. A typical application is in the field of image segmentation [16], for an overview over different graph theoretical approaches to clustering and segmentation see [24]. Also regular pyramids fit into this general framework: Their survivors are predetermined and form a regular pattern. Regular pyramids suffer from the rigidity of their structure that causes sensitivity to pixel shifts and artefacts when used for segmentation [3] or for the analysis of line drawings [11]. The abandonned regularity constraints in irregular pyramids allow random selections as used in stochastic pyramids [15], but also very sophisticated methods that adapt the new structure to the data such as adaptive pyramids [6]. But one could also imagine selection criteria that are influenced by a certain processing goal. Our approach decouples selection and contraction by clearly specifying the decimation parameters that control the reduction and by requiring a few constraints that these parameters should satisfy (see Section 3.1).

The second aspect allows several variations. Rosenfeld [19] has related parallel, degree-preserving graph contraction to multiresolution techniques. The framework he presents for parallel contraction operations depends on algebraic properties of regular graphs like trees, hypercubes, arrays, etc. Our theory extends the scope of parallel, degree-preserving graph contraction to irregular topologies. We define "connecting paths" that relate the edges of the reduced graph with paths between surviving vertices in the level below. The basic operation that contracts the graphs either step-by-step or in a few parallel steps is dual contraction. It contracts one edge and its two endpoints into one single vertex and removes the corresponding dual edge. The contraction of a graph reduces the number of vertices while maintaining the connections to other vertices. As a consequence self-loops and double edges may occure. The elimination of such non-simple connections may lead to configurations that corrupt the connectivity structure given in the input graph. We shall overcome these problems by considering the dual graph.

The remainder of this paper is organized as follows. Section 2 recapitulates the basic notions from graph theory and introduces the concept of dual image graphs. Considering crossing of paths and interior vertices we define the structure of a graph. Based on this framework, we define what we mean by a structure preserving contraction (Section 3). Dual graph contraction proceeds in two phases, dual edge contraction and dual face contraction. Both of these two operations are defined and their respective properties stated and proved in subsections 3.2 and 3.3 respectively. The introduced concepts are illustrated by means of simple examples. Section 4 compares the structural properties of three related ways to build irregular pyramids. The conclusion summarizes the results, offers several possibilities for selecting the decimation parameters and for reducing the information stored in the cells of the pyramid. Some remarks about extensions of the concept conclude the paper.

2 Dual image graphs and their structure

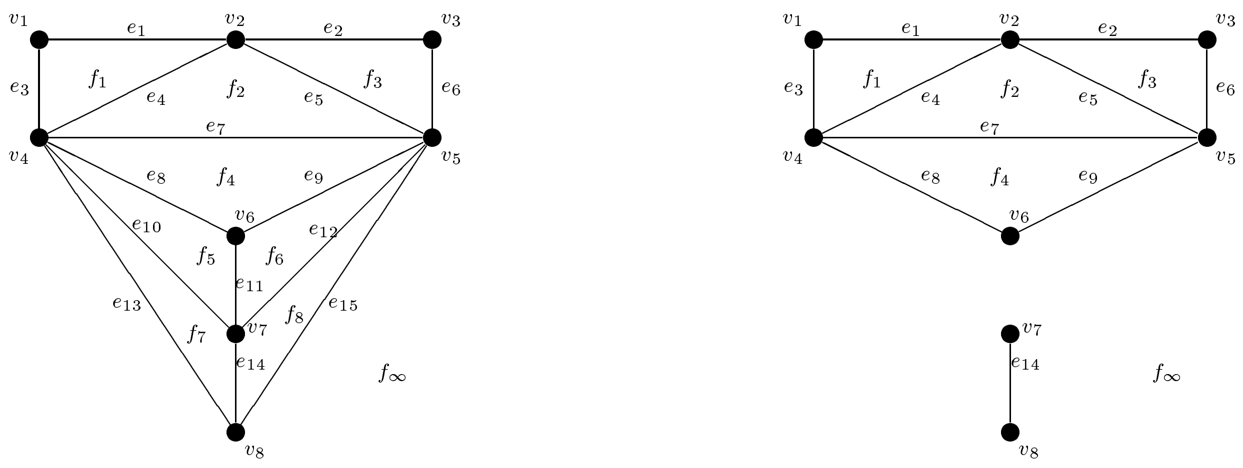
This section assembles the terminology from graph theory that is needed to define the type of graphs and the notations that describe a structure in a digital image.

We use *graphs* $G(V, E)$ consisting of *vertices* $v \in V$ and (non-directed) *edges* $e \in E$. An edge e connects two vertices v, w, $e_i = (v, w)$, an edge with $v = w$ is called a *self-loop*: $e_i(v, v)$. A graph may contain more than one edge between the same end vertices (i.e. $e'_3(v'_1, v_4) \neq e'_4(v'_1, v_4)$ in Fig. 6a), they are called *double edges*[1]. Edges are uniquely identified by indices. The *degree* of a vertex v, $\deg(v)$, is the number of edges *incident* on it. A vertex $v \in V$ is *isolated* if it has degree 0, i.e. $\deg(v) = 0$. Formal definitions of standard notions are taken from [21, 5], here, a simple example explains the basic terms.

Figure 1 shows a graph $G_1(V_1, E_1)$, with vertices $V_1 = \{v_1, \ldots, v_8\}$ and edges $E_1 = \{e_1, \ldots, e_{15}\}$. Edge $e_1(v_1, v_2)$ connects vertices v_1 and v_2. The degree of vertex v_5 is six, e.g. $\deg(v_5) = 6$, since the six edges $e_5, e_6, e_7, e_9, e_{12}, e_{15}$ are incident to v_5. *Path* $P_{63}(v_6, v_3) = (v_6, e_{11}, v_7, e_{12}, v_5, e_6, v_3)$ connects v_6 with v_3 traversing three edges. It has *length* three, $\|P_{63}\| = 3$, the same length as path $P_{18}(v_1, v_8) = (v_1, e_1, v_2, e_5, v_5, e_{15}, v_8)$. The *circuit* $C_1 = (v_8, e_{13}, v_4, e_7, v_5, e_{15}, v_8)$ in Fig. 1(a) is a closed path in G_1. Since any pair of vertices of G_1 can be connected by a path in G_1, graph G_1 is *connected*. If edges are removed from E_1, the graph may become disconnected. After removal of $E_c = \{e_{10}, e_{11}, e_{12}, e_{13}, e_{15}\}$, graph $G'_1(V_1, E_1 \setminus E_c)$ is disconnected (Fig. 1(b)) and consists of two *connected components* $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\{v_7, v_8\}$. The subset of edges $E_c \subset E_1$ is called a *cutset*.

Graph $G_1(V_1, E_1)$ is *planar* since it is drawn in the plane without any edge crossing another edge. A graph can be embedded in the plane in many ways. A graph already embedded in the plane is called a *plane graph*. The planar embedding of G_1 in Fig. 1(a)

[1] Another name is *parallel edge*.



(a) Graph $G_1(V_1, E_1)$ before and

(b) after removal of cutset.

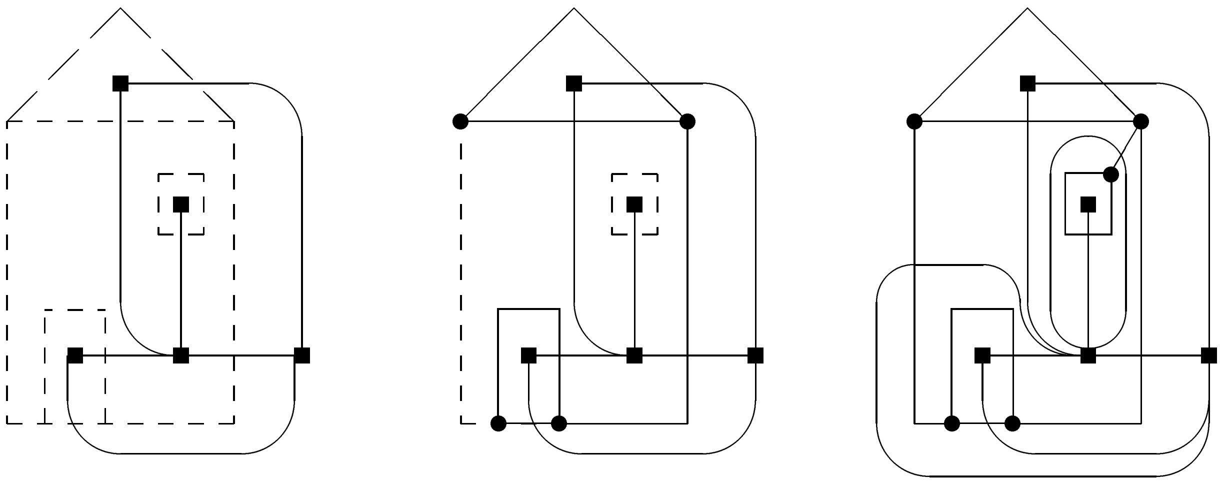
Figure 1: Graph $G_1(V_1, E_1)$ is disconnected by cutset $\{e_{10}, e_{11}, e_{12}, e_{13}, e_{15}\}$.

divides the plane into 8 (finite) regions which are called *faces*, $f_1, \ldots, f_8$, and one infinite region, the *background face* f_∞. A *cycle* $C(f)$ delimits exactly one face f, e.g. $C(f_3) = (v_2, e_5, v_5, e_6, v_3, e_2, v_2)$. The *boundary* of a (finite) graph is the cycle delimiting the background face, $C_\infty := C(f_\infty) = (v_1, e_1, v_2, e_2, v_3, e_6, v_5, e_{15},\ v_8, e_{13}, v_4, e_3, v_1)$. The adjacency of the faces in G_1 is expressed by the dual graph, $\overline{G_1}(\overline{V_1}, \overline{E_1})$, Fig. 10(b). There exists a one-to-one correspondence between the edges $\overline{e_i}$ of $\overline{G_1}$ and the edges e_i of G_1. Furthermore, any set of edges is a circuit in $\overline{G_1}$ if and only if the corresponding set of edges is a cutset in G_1. E.g. the edges corresponding to cutset $E_c \subset E_1$ form a circuit $(\overline{v_7}, \overline{e_{10}}, \overline{v_5}, \overline{e_{11}}, \overline{v_6}, \overline{e_{12}}, \overline{v_8}, \overline{e_{15}}, \overline{v_\infty}, \overline{e_{13}}, \overline{v_7})$ in $\overline{G_1}$.

2.1 Graphs of images

Our graphs describe the neighborhood relations in a digital image. At low level processing, a pixel of the sensor array is associated with a vertex and pixels adjacent either in a row or in a column are joined by an edge (note that we use 4-connectivity). The gray value or any more complex description is considered as an attribute of a vertex but is not directly used in the algorithms of this paper. The resulting graphs have several properties, they are finite, connected, and plane. We consider both the *neighborhood graph* $G(V, E)$ and its dual graph $\overline{G}(\overline{V}, \overline{E})$ in parallel. Since the vertices of $\overline{G}$ are the faces of G we refer to $\overline{G}$ as the *face graph*. This pair of related graphs is the basis of all further considerations.

The same graph formalism as for the pixel array can be used also at intermediate levels of image analysis: Region adjacency graphs (RAGs) are the result of segmentation processes. Regions are connected sets of pixels, two regions are separated by the common boundaries. Although RAGs are connected since the regions partition the image plane, their geometric duals may cause problems. Consider the RAG, $\overline{G_2}(\{\blacksquare\}, \{\blacksquare - -\blacksquare\})$, of the



(a) RAG $\overline{G_2}$

(b) $G_2 = dual(\overline{G_2})$

(c) DIG $(G_2', \overline{G_2'})$

Figure 2: A house: (a) RAG $\overline{G_2}$; (b) reconstructed G_2; (c) corrected DIG $(G_2', \overline{G_2'})$.

house example in Fig. 2(a). The five regions of the house, e.g. roof, window, door, front side, and background, are indicated by dashed lines. To reconstruct the boundary graph G_2, i.e. the dual of $\overline{G_2}$, we insert a vertex ($\bullet$) in each region of $\overline{G_2}$ and place them on the dashed boundary, preferably at boundary intersections. Then we draw the edges of E_2 by following the dashed boundary lines until crossing an edge of $\overline{E_2}$ (similar to [5][p.113]). Two problems arise in this case:

1. The window is completely surrounded by the region of the front side. Hence its boundary is not connected with the boundary of the front side. Where to place the vertex of V_2? If placed as shown in Fig. 2(b) the above algorithm terminates but does not find any edge crossing the window boundary. In the other placement the algorithm does not find any correct solution.

2. The left hand boundary of the front side is not crossed by any edge of $\overline{E_2}$.

The problems are caused by the fact that the front side's boundary consists of two non-connected pieces: the inner piece common with the window, and the outer piece being further split into four segments: one segment separates it from the roof, another from the door, and two distinct segments separate it from the background. In fact graph $\overline{G_2}$ does not express that the window is completely within the front side and that the door creates the two distinct boundary segments separating it from the background. A solution is shown in Fig. 2(c): a self-loop around the window is added in $\overline{G_2'}$, front side and background are connected by a double edge in $\overline{E_2'}$, and a 'bridge' edge in G_2' connects the boundary of the window with the boundary of the front side. The resulting pair of graphs are connected and plane, but, unfortunately, in general not simple. E.g. they may contain self-loops and double edges. However not all possible self-loops and double edges are necessary. The necessary cases can be limited to those where the self-loop or the double edges enclose non-neglectable details like the window or the door in the above example. Redundant configurations will be characterized by degenerated vertices in the dual graph (section 3.3). The following definition summarizes the properties of dual image graphs.

Definition 1 (Dual Image Graphs) *The graphs* $(G(V,E), \overline{G}(\overline{V}, \overline{E}))$ *are called* **dual image graphs** *(DIGs) if they have the following properties:*

- *both* G *and* $\overline{G}$ *are finite;*
- *both* G *and* $\overline{G}$ *are connected;*
- *both* G *and* $\overline{G}$ *are plane;*
- $\overline{G}$ *is the dual of* G*;*
- *both* G *and* $\overline{G}$ *need not be simple in general.*

2.2 The structure of plane graphs

The structure of an image plays a fundamental role in image analysis because it is invariant to any 2D image transformation and because it allows to identify objects in images by their topological structure. But what do we mean by structure precisely? We have encountered already several properties that characterize a structure and that allow to disambiguate different structures.

The two paths P_{63} and P_{18} in graph G_1 of Fig. 1(a) *intersect* at vertex v_5. More formally we define whether two paths cross each other in a given graph.

Definition 2 (Crossing Paths) *Let* P_1 *and* P_2 *be two paths in a plane graph* $G(V,E)$ *with a common path* $P_0 \subset P_1 \cap P_2$*, such that* $P_1 = (P_{1a}, P_0, P_{1b})$ *and* $P_2 = (P_{2a}, P_0, P_{2b})$*.* P_0 *can be as short as only one single vertex. Path* P_1 *crosses path* P_2 *if the four path tails alternate in a clockwise enumeration around* P_0*, e.g.* $(P_{1a}, P_{2a}, P_{1b}, P_{2b})$ *(Fig. 3).*

A substructure like a single vertex, a single face, a subgraph, ... that is completely surrounded by a circuit contributes also to the structure. Remember the window in the house example. In Fig. 1(a) circuit $C_1 = (v_8, e_{13}, v_4, e_7, v_5, e_{15}, v_8)$ completely surrounds vertex v_6. We call v_6 *interior vertex* of C_1 and define this relation between a single vertex and a circuit in a plane graph as follows:

Definition 3 (Interior Vertex) *Let* $C \neq C_\infty$ *be a circuit in a finite, connected, plane graph* $G(V,E)$*. Furthermore, let* C_∞ *denote the cycle delimiting the background of* G*. A vertex* $v \in V$ *is called an* **interior vertex of** C *if there is no path* $P(v, v_\infty)$ *connecting* v *to any vertex* $v_\infty \in C_\infty$ *without crossing* C*, e.g.* $P(v, v_\infty) \cap C \neq \emptyset$ *(Fig. 4). Circuit* C *is said to 'surround' vertex* v*.*

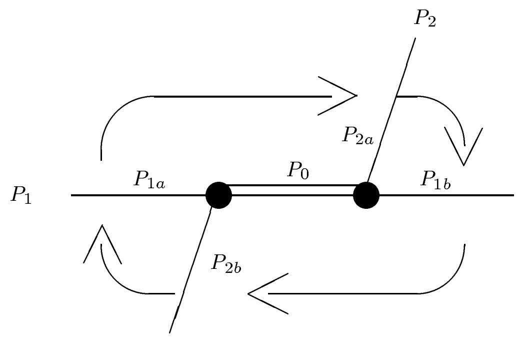


Figure 3: Crossing of two paths P_1 and P_2.

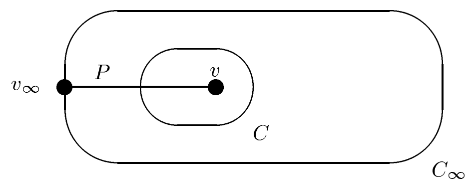


Figure 4: Vertex v is interior of circuit C.

We describe an image's adjacency relations by a pair of plane graphs. The formal definition of the structure of a plane graph collects all the above determining factors.

Definition 4 (Structure of a Plane Graph) *Let $G(V, E)$ be a finite, connected, plane graph. Furthermore, let $S_G(v)$ denote the family of all circuits surrounding vertex $v \in V$ in graph G. Then we define as the structure of G the following set:*

$$Struct(G) := \{(v, S_G(v)) | v \in V\}$$

3 Dual Graph Contraction

In this section we present the algorithm that simplifies the structure of a pair of dual image graphs. The contraction process is controlled by decimation parameters. Selected subsets of vertices and of edges of the original neighborhood graph define the relation between the contracted and the original graphs. Subsection 3.1 specifies the required properties of the contracted graphs. The structure modification consists of two elementary operations described in subsections 3.2 and 3.3 that are combined in the algorithm in subsection 3.4.

3.1 Structure preserving contraction

Stochastic decimation as proposed by Meer [15] is controlled by selecting surviving and non-surviving vertices, and by defining receptive fields that completely cover the input data. Jolion and Montanvert [6] showed how this selection must be modified such that decimation is controlled by the image data in order to achieve an adaptive behavior of the process. Another interesting approach proposed by Bischof [2] is to control the extraction of the set of survivors by an energy term of the kind used in Hopfield neural networks.

Definition 5 (Decimation Parameters) *Consider a graph $G(V, E)$. A decimation of graph G is specified by a selection of* **surviving** *vertices $V_s \subset V$ and a selection of a subset E_{sn} of edges E. The sets (V_s, E_{sn}) are called* **decimation parameters**. *We call $V_n := V \setminus V_s$* **non-surviving vertices**. *E_{sn} must be a subset of $(V_s \times V_n) \cap E$ and it connects all non-surviving vertices to exactly one surviving vertex in a unique way:*

$$\forall v_n \in V_n \qquad \exists! v_s \in V_s \qquad \exists! e \in E_{sn} \qquad e = (v_s, v_n) \tag{1}$$

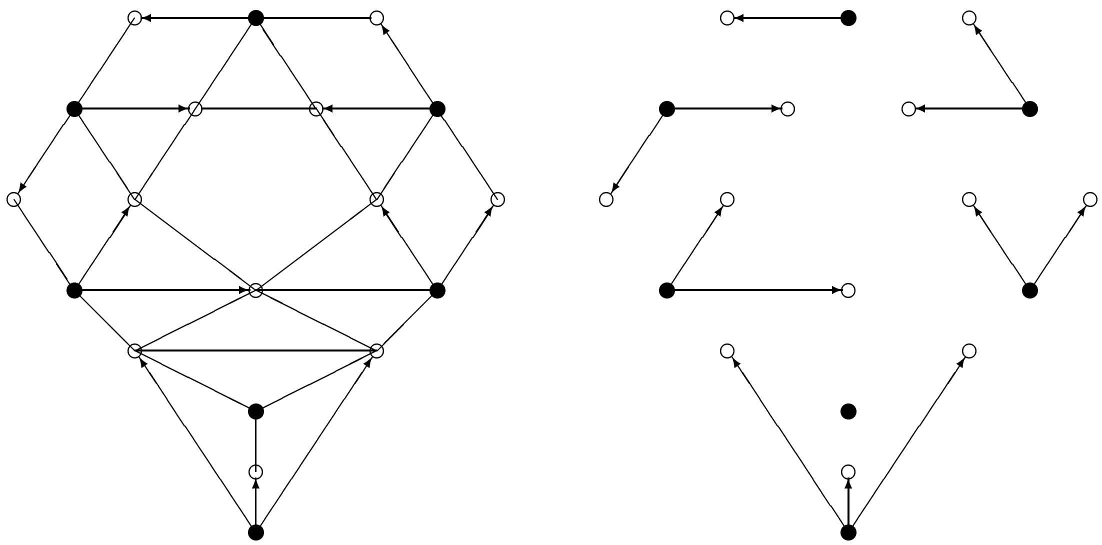


Figure 5: Decimation of $G_3(V_3, E_3)$ creates trees in graph (V_3, E_{sn}).

Subgraph (V, E_{sn}) partitions G into the same number of connected components as there are surviving vertices in V_s. Each component forms a tree structure connecting the surviving vertex, the **parent** ($\bullet$), to the non-surviving vertices, the **children** ($\circ$), by edges of E_{sn} ($\bullet \rightarrow \circ$, see example in Fig. 5).

Note that our definition does not constrain the selection of surviving vertices, as does the requirement that they must form a maximum independent set (MIS) in stochastic pyramids [15]. The only condition that our decimation parameters must satisfy requires that at least one of the neighbors of a non-surviving vertex must survive.

Before defining the properties that characterize a contracted graph $G'(V', E')$ we introduce *connecting paths* in $G(V, E)$ that relate edges of E' with paths in G.

Definition 6 (Connecting Path) *Let G(V,E) be a graph with decimation parameters (V_s, E_{sn}). A path in $G(V, E)$ is called a* **connecting path** *of two surviving vertices $v_b, v_e \in V_s$, denoted $CP(v_b, v_e)$, if one of the following conditions is satisfied:*

1. *v_b and v_e are connected by an edge e_{be} in G: $CP(v_b, v_e) = (v_b, e_{be}, v_e); e_{be} \in E$.*
2. *The path contains two edges, $CP(v_b, v_e) = (v_b, e_{bi}, v_i, e_{ie}, v_e)$ with $v_i \in V_n$ and one of the two edges is in E_{sn}.*
3. *The path contains three edges of E, $CP(v_b, v_e) = (v_b, e_{bi}, v_i, e_{ij}, v_j, e_{je}, v_e)$ with both $v_i, v_j \in V_n$ and both edges $e_{bi}, e_{je} \in E_{sn}$.*

Connecting paths have lengths 1, 2, or 3. The end points of connecting paths are surviving vertices. Every connecting path contains exactly one edge that is not in E_{sn}. Connecting paths are the basis to define neighbors in the contracted graph.

Definition 7 (Structure Preserving Contraction) *Graph $G'(V', E')$ is a* **structure preserving contraction** *of a connected, plane graph $G(V, E)$ controlled by decimation parameters (V_s, E_{sn}) if following conditions are satisfied:*

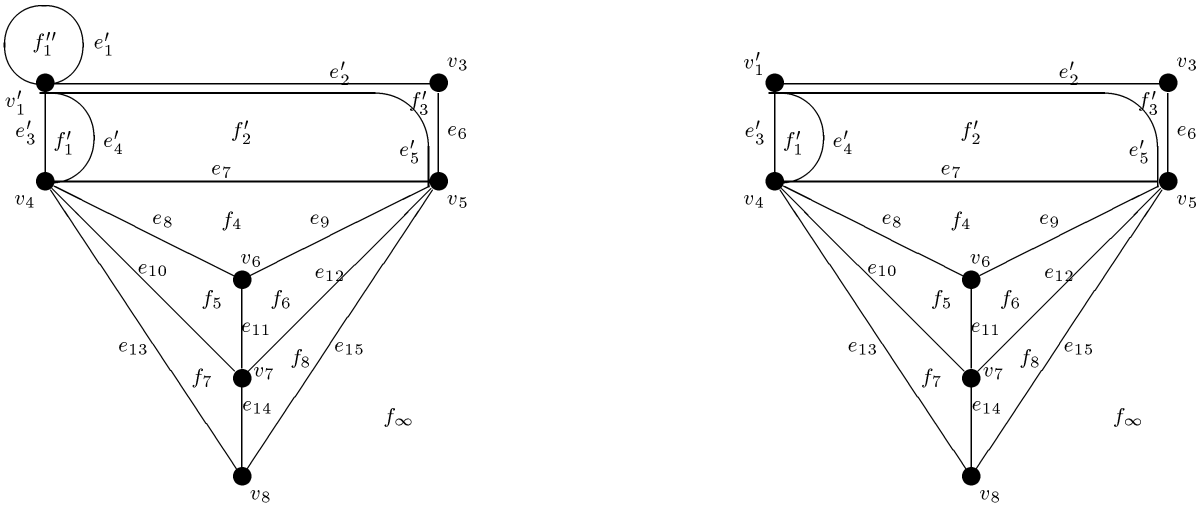
1. *$V' = V_s$.*
2. *For all edges $e' = (v_b, v_e) \in E'$ there exists a connecting path $CP(v_b, v_e)$ in G.*
3. *If $CP(v_b, v_e)$ is a connecting path in G then $v_b = v_e$ or $(v_b, v_e) \in E'$.*
4. *Let C be any sequence of connecting paths $CP(v_0, v_1), CP(v_1, v_2), \ldots, CP(v_n, v_0)$ in G forming a circuit. If there exist surviving vertices interior of C they must also be interior of the circuit $C' = (v_0, (v_0, v_1), v_1, \ldots (v_n, v_0), v_0)$ in G'.*

The first three conditions establish the correspondence between graph $G(V, E)$ and the contracted graph $G'(V', E')$. The selected survivors V_s are the vertices of the contracted graph V'. Edges in E' correspond to connecting paths in G and vice versa, and, consequently, circuits in G' have corresponding circuits in G. Circuit C in the fourth condition characterizes all circuits in G that have a corresponding circuit in G'. Let $v_s \in V_s$ be surrounded by C, then $C \in S_G(v_s)$ (cf. Def.4). Condition 4 requires that any 'surviving' part (v_s, C) of the structure of G is preserved in the structure of G', e.g. $C' \in S_{G'}(v_s)$. Since this must be true for all circuits C' surrounding v_s in G', $(v_s, S_{G'}(v_s)) \in Struct(G')$.

3.2 Dual contraction of non-surviving vertices

Two vertices v_i and v_j in a graph $G(V, E)$ are **identified** by replacing both vertices by a new vertex which is connected to all vertices that were incident on v_i and v_j before identification.

Definition 8 (Edge Contraction) Contraction *of an edge* $e \in E$ *in a graph* $G(V, E)$ *is the operation of removing* e *from* E *and identifying its end vertices.*



(a) **Identification** of v_1 and v_2.

(b) **Contraction** of e_1.

Figure 6: Identification and contraction in $G_1(V_1, E_1)$.

Figure 6 illustrates the difference between identification and contraction for the example of the graph in Figure 1. The edge e_1' forms a self-loop at the new vertex v_1' after vertices v_1 and v_2 are identified, whereas contraction eliminates e_1. Note that in both cases e_3' and e_4' become double edges.

Definition 9 (Dual Edge Contraction) *Let* $G(V, E)$ *and* $\overline{G}(\overline{V}, \overline{E})$ *be dual image graphs.* **Dual contraction** *contracts an edge* $e \in E$ *and removes its corresponding edge* $\overline{e} \in \overline{E}$ *from* $\overline{G}$ *at the same time.*

Theorem 1 *Let* $G(V, E)$ *and* $\overline{G}(\overline{V}, \overline{E})$ *denote dual image graphs and* (V_s, E_{sn}) *the decimation parameters. Dually contracting all edges of* E_{sn} *collapses all non-surviving vertices into their surviving parents and creates a contracted graph* $G'(V', E')$ *that preserves the structure of* $G(V, E)$ *(according to Def. 7). All connecting paths become edges of the contracted graph* $G'(V', E')$ *connecting the surviving endpoints.*

Proof : We show that $G'(V', E')$ satisfies the four properties of a structure preserving contraction (Def. 7):

1. Since all non-surviving vertices of V_n are connected to exactly one surviving vertex of V_s by exactly one edge in E_{sn} (Def. 5) and since all edges of E_{sn} have been contraced to their surviving vertex, $V' \cap V_n = \emptyset$. Since contraction does not remove any surviving vertex, $V_s \subset V' \subset V = V_s \cup V_n$, and, hence, $V' = V_s$.

2. Let $e' = (v_b, v_e) \in E'$ then both $v_b \in V_s$ and $v_e \in V_s$. By construction an edge $e' \in E'$ is either also an edge in E or the result of contraction. In the first case (v_b, e', v_e) is a connecting path.

   If e' is the result of contracting only one edge there must be a path $(v_b, e_{bi}, v_i, e_{ie}, v_e)$ connecting v_b and v_e in G. Either $e_{bi} = (v_b, v_i) \in E_{sn}$ and $e_{ie} = (v_i, v_e) \in E$ or $e_{bi} = (v_b, v_i) \in E$ and $e_{ie} = (v_i, v_e) \in E_{sn}$. In both cases the path is a connecting path.

   Edge e' could also be the result of two contractions originating from a path $(v_b, e_{bi}, v_i, e_{ij}, v_j, e_{je}, v_e)$. In this case $v_i, v_j \in V_n$, $e_{bi}, e_{je} \in E_{sn}$, and $e_{ij} \in E$. Again it is a connecting path.

   Longer paths cannot result in a single (contracted) edge since the path connects two surviving vertices with a sequence of non-surviving vertices. The middle of three non-surviving vertices must have a parent v_m other than v_b, v_e. Otherwise there would be a shorter path connecting v_b to v_e. After contracting all edges in E_{sn}, (v_b, v_e) cannot be in E' unless there is another (shorter) connecting path.

3. Let $CP(v_b, v_e)$ be a connecting path in G, $v_b, v_e \in V_s$. We distinguish the three cases of Def. 6:

   (a) $CP(v_b, v_e) = (v_b, e_{be}, v_e)$; $e_{be} \in E$: no contraction takes place, $e_{be} \in E'$.

   (b) $CP(v_b, v_e) = (v_b, e_{bi}, v_i, e_{ie}, v_e)$ with $v_i \in V_n$ and one of the two edges in E_{sn}: the edge in E_{sn} is contracted, v_i is identified with either v_b or v_e, and the remaining edge connects v_b and v_e, e.g. $(v_b, v_e) \in E'$.

   (c) $CP(v_b, v_e) = (v_b, e_{bi}, v_i, e_{ij}, v_j, e_{je}, v_e)$ with both $v_i, v_j \in V_n$ and both edges $e_{bi}, e_{je} \in E_{sn}$: the contraction removes the two edges of E_{sn} and removes the non-surviving vertices v_i, v_j, the remaining edge connects v_b and v_e, e.g. $(v_b, v_e) \in E'$.

4. Assume C is a circuit surrounding $v_{n+1} \in V_s$ in G, $C = CP(v_0, v_1), CP(v_1, v_2), \ldots, CP(v_n, v_0)$ (Fig. 7). Then all paths $P(v_{n+1}, v_\infty)$ must intersect C (acc. to Def. 3).

   Before starting the indirect proof we check whether $C' \in G'$ has the necessary properties. $C' = (v_0, (v_0, v_1), v_1, \ldots (v_n, v_0), v_0)$ is a circuit in G' because connecting paths of C are contracted to edges in E' and because C was a circuit in G. Graph G' is plane because we can embed all edges of G' using polygons spanned by the corresponding

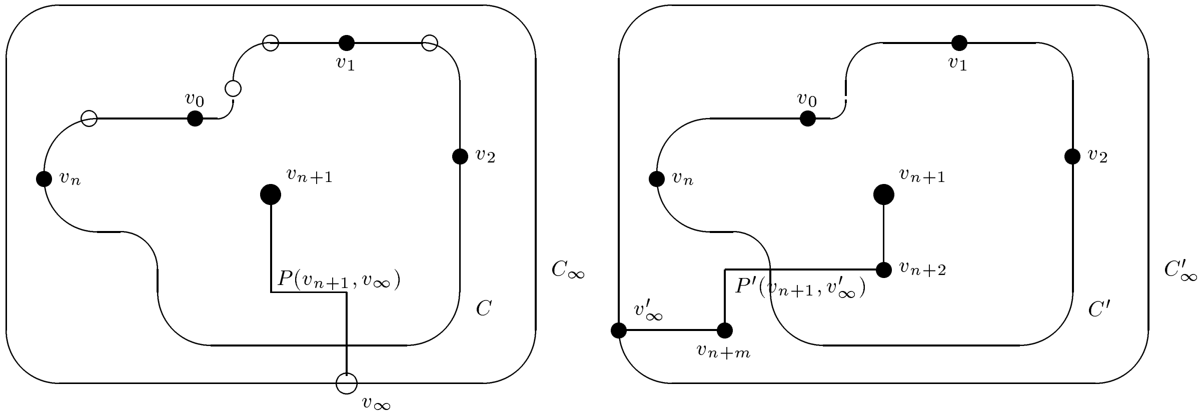


Figure 7: v_{n+1} interior to C remains interior to C' in G'.

connecting paths which are embedded in the plane graph G by assumption (for more details see [14]). Finally, G' is connected if G is connected since contraction preserves connectivity.

The indirect proof assumes that v_{n+1} is not surrounded by C' in G' and shows that all refinements of the path $P'(v_{n+1}, v'_\infty)$ in G contradict the fact that v_{n+1} is interior to C. Assume that there exists a path $P'(v_{n+1}, v'_\infty)$ in G' such that $P'(v_{n+1}, v'_\infty) \cap C' = \emptyset$. This path consists of edges $(v_{n+1}, v_{n+2}), \ldots (v_{n+m}, v'_\infty) \in E'$. The corresponding connecting paths $CP(v_{n+j}, v_{n+j+1}), j = 1, \ldots, m-1$, and $CP(v_{n+m}, v'_\infty)$ form a path $P_0(v_{n+1}, v'_\infty)$ in G that connects v_{n+1} with v'_∞.

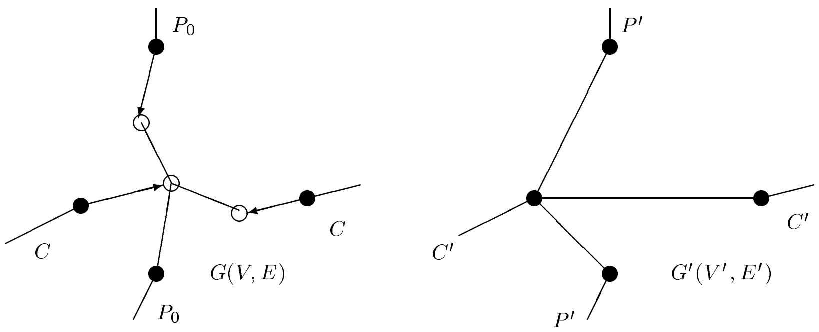


Figure 8: Paths crossing in $G : C \cap P_0$ and in $G' : P' \cap C'$.

By construction of path P_0 the intersection with C across surviving vertices $v_{n+j} \in V_s; j = 1, \ldots, m$ is impossible, hence $P_0 \cap C \subset V_n$. However a non-surviving vertex has only one parent which would create an intersection of P' and C' after contraction, which is in contradiction to our assumption (Fig. 8). Therefore $P_0 \cap C = \emptyset$. If $v'_\infty \in C_\infty$, P_0 joins v_{n+1} with the background, which contradicts the first assumption that v_{n+1} is interior to C. If v'_∞ is not on the background's boundary, it must have a child v_∞ that is on the boundary. v_∞ cannot be on C (that is the only remaining case where C could surround v_{n+1}) because contraction would map it onto v'_∞ which is not on C'. Hence $P_0(v_{n+1}, v'_\infty)$ can be extended to v_∞ without intersecting C, contradiction. ■

The above process can be implemented in parallel for two reasons: (1) because the removal of edges E_{sn} from E and $\overline{E_{sn}}$ from $\overline{E}$ is independent of each other and (2) because identification simply renames all children to their parents' name in the remaining sets E and $\overline{E}$.

3.3 Dual contraction of redundant faces

Dual edge contraction of graph $G(V, E)$ decreases the number of edges in $\overline{E}$ and, hence, also the degrees of the vertices in $\overline{G}$. **Degenerated faces** with degree one and two may result. A second (dual) contraction process 'cleans' the dual graph from such degenerated faces. As a side effect many but not necessarily all of the self-loops and double edges in G are 'cleaned up' as well.

Definition 10 (Dual Face Contraction) *Consider a pair of dual image graphs $G(V, E)$ and $\overline{G}(\overline{V}, \overline{E})$. Let $\overline{v_i} \in \overline{V} \setminus \{\overline{v_\infty}\}$ be a degenerated face not being the background face,* $\deg(\overline{v_i}) < 3$, *and let $\overline{e_i}(\overline{v_i}, \overline{v_j})$ be an incident edge in $\overline{E}$. Then $\overline{e_i}$ is dually contracted, identifying $\overline{v_i}$ with $\overline{v_j}$, and eliminating edge $e_i \in E$ corresponding to $\overline{e_i}$. Since vertices of $\overline{G}$ correspond to faces of G, we refer to this process as* **dual face contraction**.

Theorem 2 *Let $G(V, E)$ and $\overline{G}(\overline{V}, \overline{E})$ be a pair of dual image graphs. Dual face contraction preserves the structure of graph $G(V, E)$.*

Proof : The two cases $\deg(\overline{v_i}) = 1$ and $\deg(\overline{v_i}) = 2$ (see Fig. 9) are discussed separately:

1. If $\deg(\overline{v_i}) = 1$ then the edge $e_i \in E$ corresponding to $\overline{e_i} = (\overline{v_i}, \overline{v_j})$ is the only edge in the circuit surrounding face f_i, i.e. $e_i = (v, v)$ is a self-loop in G. Clearly the removal of a self-loop does not disconnect G. Self-loops in G that contain interior vertices are not removed because any interior non-isolated vertex would increase $\deg(\overline{v_i}) > 1$.

2. Let $\deg(\overline{v_i}) = 2$, $\overline{e_i}, \overline{e_j} \in \overline{E}$ being the two edges incident to $\overline{v_i}$. If $\overline{e_i} = \overline{e_j}$, $\overline{e_i} = (\overline{v_i}, \overline{v_i})$ is a self-loop in $\overline{G}$. Since $\overline{G}$ is connected, $\overline{E} = \{\overline{e_i}\}$ and $\overline{V} = \{\overline{v_i}\}$, $\overline{v_i}$ being the only face. Hence $\overline{v_i} = \overline{v_\infty}$ is the background face which is excluded from face contraction. Therefore $\overline{e_i} \neq \overline{e_j}$ are different edges in $\overline{G}$ and, by duality, $e_i \neq e_j$ are different edges

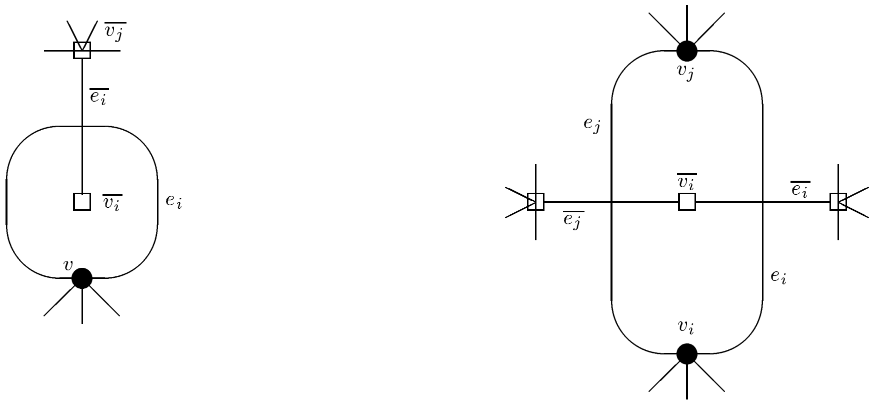


Figure 9: Face contraction of degenerated faces.

in G. Face $\overline{v_i}$ is surrounded by a circuit C with two edges in G: $C = (v_i, e_i, v_j, e_j, v_i)$. Obviously both e_i and e_j connect the same vertices v_i and v_j in G. The removal of one of such double edges preserves the connectivity of G. C does not contain any interior vertex and any other circuit surrounds the same vertices before and after contracting $\overline{v_i}$. ∎

Note that the contraction of a face may lead to another degenerated face. Furthermore, not all degenerated faces can be contracted in parallel. However a process similar to stochastic decimation can determine an independent set of degenerated faces which could be contracted in parallel. For the remaining degenerated faces the process is repeated until no further degenerated face exists in $\overline{G}$.

3.4 Combining the elementary processes

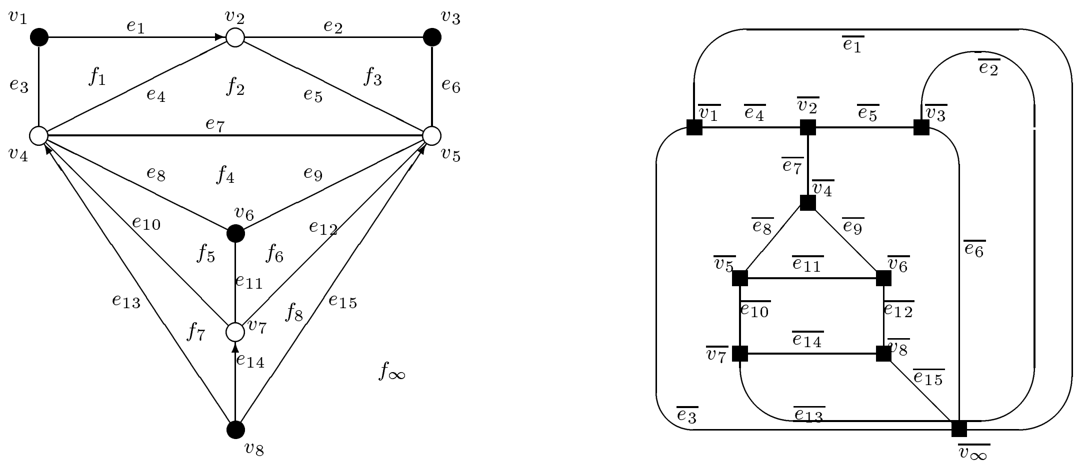
In the previous subsections, we have gathered all subprocesses we need to define the process of dual graph contraction.

Definition 11 (Dual Graph Contraction) *Let $G(V, E)$ and $\overline{G}(\overline{V}, \overline{E})$ be a pair of dual image graphs. Given the decimation parameters (V_s, E_{sn})* **dual graph contraction** *consists of the following sequence of processes applied to this pair of graphs:*

1. *Dually contract all edges $e \in E_{sn}$ collapsing all non-surviving vertices into their surviving parent vertex;*

2. *dually (face) contract all degenerated faces;*

3. *repeat step 2 until all degenerated faces have been eliminated.*



(a) G_1 with decimation parameters

(b) dual graph $\overline{G_1}$

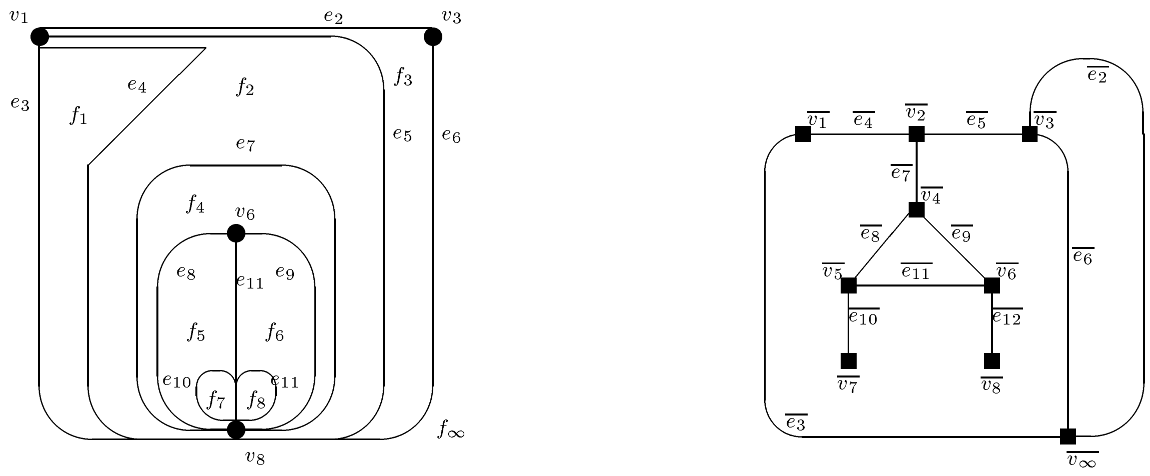
Figure 10: G_1 and $\overline{G_1}$ before dual graph contraction.

Figures 10, 11 and 12 illustrate dual graph contraction. Figure 10 shows a planar embedding of graph $G_1(V_1, E_1)$ consisting of 8 vertices and 15 edges. The plane is divided into 9 faces, with face f_∞ being the background face. Figure 10b) shows the dual of G_1, $\overline{G_1}(\overline{V_1}, \overline{E_1})$, with one vertex representing every face of G_1. Note that all finite faces form triangles, or equivalently, $\deg(\overline{v_i}) = 3, i = 1, \ldots, 8$. Figure 10a) illustrates the decimation parameters graphically: survivors $V_s = \{\bullet\}$, non-survivors $V_n = \{\circ\}$, and $E_{sn} = \{\bullet \rightarrow \circ\}$ $= \{e_1, e_{13}, e_{14}, e_{15}\}$. The result of dually contracting all edges E_{sn} is depicted in Figure 11: all parent-child connections, i.e. all edges that are drawn as arrows in Figure 10a), have been dually contracted. Graph [2] G_1^* contains three self-loops: e_7, e_{10}, e_{11}. There are three edges connecting the same two end vertices v_1 and v_8: e_3, e_4, e_5; and also three edges connecting v_6 and v_8: e_8, e_9, e_{11}. Fig. 12 results from dually contracting all degenerated faces in $\overline{G_1^*}$ successively: f_7, f_8, f_1, f_5, f_6. Note that v_6 is interior both to circuit $(v_1, e_1, v_2, e_5, v_5, e_{15}, v_8, e_{13}, v_4, e_3, v_1)$ in G which becomes $(v_1, e_5, v_8, e_3, v_1)$ in G', and to $(v_8, e_{13}, v_4, e_7, v_5, e_{15}, v_8)$ in G becoming a self-loop (v_8, e_7, v_8) in G'. Hence self-loop e_7 as well as double edge e_3, e_5 must survive to satisfy condition (4) of Def. 5.

Theorem 3 *Let $(G(V,E), \overline{G}(\overline{V}, \overline{E}))$ be a pair of dual image graphs and $(G'(V',E'), \overline{G'}(\overline{V'}, \overline{E'}))$ be the result of dual graph contraction with decimation parameters (V_s, E_{sn}). Then*

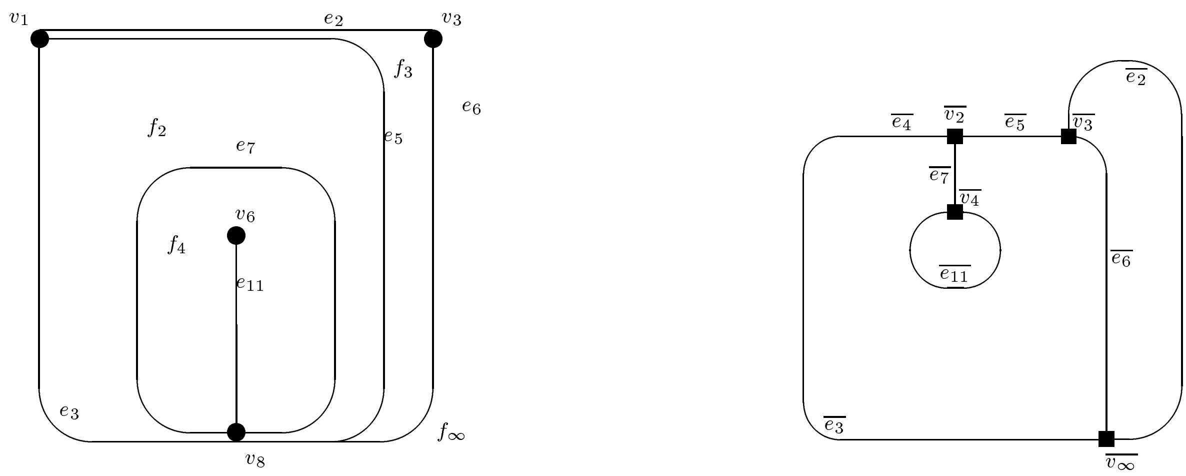
1. *$G'(V',E')$ is a structure preserving contraction of $G(V,E)$.*
2. *$(G'(V',E'), \overline{G'}(\overline{V'}, \overline{E'}))$ is minimal, i.e. no further contraction is possible.*
3. *$(G'(V',E'), \overline{G'}(\overline{V'}, \overline{E'}))$ is unique.*

[2] G^* identifies the result of dual edge contraction, G' the final result.



(a) G_1^* (b) $\overline{G_1^*}$.

Figure 11: Result of dual edge contraction: G_1^* and $\overline{G_1^*}$.



(a) G_1' (b) $\overline{G_1'}$

Figure 12: Result of dual graph contraction: G_1' and $\overline{G_1'}$.

Proof :

1. $(G'(V', E'), \overline{G'}(\overline{V'}, \overline{E'}))$ is a structure preserving contraction of $G(V, E)$ since all the involved operations, e.g. dual edge contraction and dual face contraction, preserve the structure given in $G(V, E)$, as proved in Theorems 1 and 2.

   However, connectivity of $\overline{G'}$ has not been shown yet. Connectivity of graph $\overline{G}$ is preserved when degenerated faces are contracted. Therefore, $\overline{G}$ could be disconnected only by dual edge contraction. Let $e_i \in E_{sn}$ be the last edge the dual contraction of which would split $\overline{G}$ into two components. Hence $\overline{e_i}$ is the only connection between the two parts before splitting and, as a consequence, e_i must be a self-loop in G. This contradicts the assumption that $e_i \in E_{sn}$ connects a surviving with a non-surviving vertex.

2. $(G'(V', E'), \overline{G'}(\overline{V'}, \overline{E'}))$ is minimal if any further contraction would destroy the desired properties of G'. Since $V' = V_s$, no further vertex can be removed by dual edge contraction. After step (3) of dual graph contraction there are no degenerated faces other than the background face in $\overline{G'}$, e.g. $\deg(\overline{v_i}) > 2$ for all $\overline{v_i} \in \overline{V'} \setminus \{\overline{v_\infty}\}$. Let us consider the consequences of dually contracting a face with $\deg(\overline{v_i}) > 2$ by dually contracting an incident edge $\overline{e_{ij}} = (\overline{v_i}, \overline{v_j})$. The removal of edge e_{ij} from G' would either disconnect two vertices that were connected before (and, hence, were also connected in G by a connecting path) or it would open a circuit build by a double edge that surrounds a surviving subgraph. This substructure exists because otherwise the double edge would include a degenerated face with only two sides.

3. $G'(V', E')$ is unique. It is clear that the result after step (1) is unique since it removes all non-surviving vertices and since the individual operations are independent.

   Now let us assume we have derived two different results $(G'_1(V'_1, E'_1), \overline{G'_1}(\overline{V'_1}, \overline{E'_1}))$ and $(G'_2(V'_2, E'_2), \overline{G'_2}(\overline{V'_2}, \overline{E'_2}))$ by dually contracting the faces in a different order. $V'_1 = V'_2$ because face contraction does not change vertices. The connectivity in G' is determined by the connecting paths and not by the order of face contractions. Hence also $E'_1 = E'_2$ and $G'_1 = G'_2$. The dual graphs may differ only by a different planar embedding. But this is determined by the structure of the graphs before dual contraction and preserved by dual face contraction. ■

4 Three different ways to build irregular pyramids

Def. 7 specified four properties that relate the original graph G and its contraction G'. It was argued that these conditions should preserve certain structural properties of graph G. With some slight modifications of the requirements other results can be achieved. This section compares the introduced version with two modifications. In the examples we shall use graph G_3 from Fig. 5(a) as our original graph.

The following property has been observed first in [13] but the present formulation allows a much clearer proof.

Theorem 4 *Let $(G(V,E), \overline{G}(\overline{V}, \overline{E}))$ be a pair of dual image graphs and $(G'(V', E'), \overline{G'}(\overline{V'}, \overline{E'}))$ the result of dual graph contraction. Then the degree of vertices of $\overline{G'}$ is less or equal to the degree of vertices of $\overline{G}$.*

Proof : Dual edge contraction removes dual edges in $\overline{G}$, but the number of faces remains the same as in $\overline{V}$. However the degrees of the two adjacent faces decrease by one when a dual edge is removed. Dual face contraction eliminates degenerated faces. Contraction of a face of degree one reduces the degree of the other adjacent face by one. Contraction of a face with degree two leaves the degrees of the two adjacent faces the same. Hence all faces of $\overline{G'}$ can find a face in $\overline{G}$ with at least the same degree. ■

If we relaxe the requirement to preserve structure (fourth condition in Def. 7), the resulting graphs need no self-loops nor any double edges. The minimal graph satisfying conditions (1), (2), and (3) is *simple*, it has been used in the previous works of Meer [15], Montanvert [16] and Jolion [6]. The such defined simple graph is a subgraph of a structure preserving contraction. Since the structure preserving contraction preserves planarity this is also the case for the simple graph. Let us refer to this type of contraction as *simple contraction*. The only drawback of the simple contraction is that the degrees of faces cannot be garanteed to shrink in certain cases, e.g. when there exists a vertex with degree one (see Fig. 13b).

The second modification uses a different definition for connecting paths. Connecting paths are all paths of lengths less than four in G that connect surviving vertices, resulting in following edges of the reduced graph G': $E' := \{(u,v) \in V_s \times V_s | \exists P(u,v) \in G$ such that $\|P(u,v)\| < 4\}$. Let us refer to this graph reduction as *path length contraction* although it does not necessarily involve a contraction operation. With this simplification, the selection of E_{sn} is no more necessary. In addition, planarity cannot be preserved. This is illustrated in Figure 13 which shows the result of the three different contractions of the same graph G_3 shown in Fig. 5. Although the original graph is planar the graph in Fig. 13(a) contains the complete graph K_5 as subgraph. Preliminary experiments show a property similar to the non-increasing face degrees: the lengths of cycles seem not to increase[3]. We close this section with Table 1 that compares several properties of the three contractions.

5 Conclusion

Dual graph contraction transforms a pair of dual image graphs into a pair of smaller dual image graphs. The contraction is controlled by decimation parameters. Surviving vertices can be chosen as an arbitrary subset of vertices, only non-surviving vertices must

[3]This conjecture is not proved yet

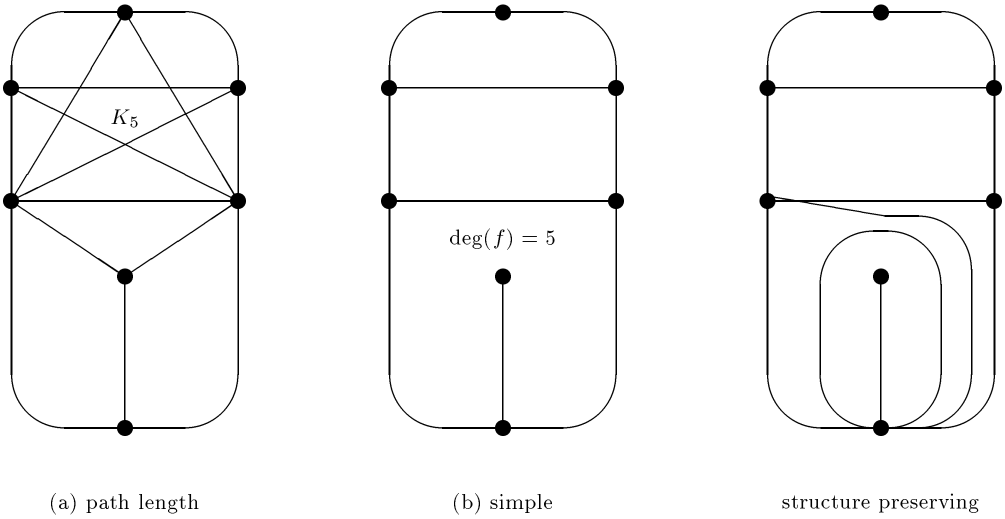


Figure 13: Three different contractions of G_3.

Table 1: Comparison of three contractions $G(V, E) \to G'(V', E')$.

path length	simple	structure preserving
$V' := V_s$	$V' := V_s$	$V' := V_s$
$\|CP(u, v)\| < 4$	$CP(u, v)$ (Def. 6)	$CP(u, v)$ (Def. 6)
$(u, v) \in E' \Leftrightarrow \exists \|P(u, v)\| < 4 \in G$	$(u, v) \in E' \Leftrightarrow CP(u, v) \in G$	(1)–(4) of Def. 7
no double edge	no double edge	some double edges
no self-loop	no self-loop	some self-loops
planar $\to$ non-planar	planar $\to$ planar	planar $\to$ planar
connected $\to$ connected	connected $\to$ connected	connected $\to$ connected
preserving lengths of cycles(?)	not preserving face degrees	preserving face degrees

satisfy a minor constraint. It is shown that the result preserves the structure given before contraction. Furthermore it fulfills all requirements for dual image graphs to be contracted again. Applied recursively, the algorithm builds an irregular pyramid.

Our experience with the different approaches for reducing graph structures and the new approach presented in this paper extends the scope of the presented theory to three and higher dimensions. We observed that contraction led to degenerations both in the original and in the dual graph (self-loops, double edges). When removed in the original graph the structure could not be preserved. However the removal of degenerations in the dual graph nicely removed all degenerations that did not distroy the structure of the graph. In 3D space duality can be introduced between points and volumes, and between lines and faces. A similar dual contraction scheme could be applied to build 3D irregular pyramids.

A further increase in dimensionality could be possible through the use of abstract cellular complexes (ACC), introduced by Steinitz [20], and introduced in the field of image analysis by Kovalevsky ([9], loc. cit. [8], and [10]). In this general theory, points are considered as space elements of dimension 0, lines are assigned dimension 1, faces dimension 2, and volumes dimension 3. In an ACC of dimension n, space elements of any dimension are bounded by space elements of all lower dimensions. This concept opens the perspective to build irregular pyramids in dimensions even higher than three. Then tasks like adaptive clustering in n-dimensional feature spaces could be computed with the efficiency of pyramidal processing.

How to select the decimation parameters has not been discussed in this paper. There are several possibilities to determine these parameters, each criterium following a different objective:

- Random selection as in Meer's stochastic pyramids [15];
- MIS determination by a Hopfield neural network [2, 1];
- Adapting the pyramid structure by data dependent local voting like in [6];
- Enforcing certain model-guided subgraph structures that could be predetermined by the vocabulary of interpretation.

Besides their structural information the vertices and edges of dual image graphs carry additional information as do the pixels of a picture array. Semantic information can be added to the graphs by attributes or labels. During contraction these attributes must be calculated also for the reduced graph. In analogy to regular pyramids reduction functions [12] serve this purpose. They take as input the attributes of all children to compute the parent's attribute. Subsampling or averaging would be simple examples. The real potential of irregular pyramids lies probably in the efficient combination of reducing information and adaptively contracting the structure.

Acknowledgments

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