Technical Report

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Dual Irregular Voronoi Pyramids and Segmentation[1]

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Abstract

We continue previous work about the combination of top-down and bottom-up adaptive segmentation techniques, Voronoi diagrams and irregular pyramids. We extend our considerations to the dual irregular pyramid to overcome the problem of increasing degree inherent to the "classical" irregular pyramid. Experimental results are presented, the analysis of which reveals inconsistencies in the theory of dual irregular pyramids. The conclusion of the report outlines two strategies for research in view of a solution.

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1 Introduction

Voronoi diagrams are useful to achieve a greylevel segmentation of a digital image in a top-down process [1, 7, 4]. Irregular pyramids can be used to do the same in a bottom-up manner [13]. We have successfully combined the two approaches in a previous work [3]. In this report we continue this research by taking into consideration the dual irregular pyramid that can solve the problem of unbounded degree that irregular pyramids suffer from.

Levels of an irregular pyramid are general graph structures. In applications of segmentation, vertices of these graphs represent regions of a segmented image, i. e. the structure of the pyramid can reflect the structure of the scene represented by the image data. However, the size of a vertex' neighborhood, its degree, is not bounded from level to level, and, consequently, neither are both time complexity of local processes and memory space required for representation [11].

Using the concept of dual graphs can avoid the problem of non-bounded size of neighborhood present in irregular pyramids. The main difference to the "classical" irregular pyramid is the extension of the set of space elements that are used to define the topology of the pyramid levels. In addition to zero- and one-dimensional space elements (vertices and edges) considered by Meer [12] in his proposition of the irregular pyramid concept, we have introduced two dimensional space elements (faces) into our considerations, and we represent them as vertices of the dual graph. The process of decimation as described in [12] divides the vertices of a level of the pyramid into survivors and non-survivors. To do the same for the set of faces, we required surviving faces that are spanned by at least three vertices [11].

In this technical report we first briefly review Voronoi diagrams and irregular pyramids in Sections 2 and 3, and then introduce the concept of duality that helps to overcome the problem of increasing degree in irregular pyramids (Section 4). In Section 5 we present the example of a dual irregular pyramid built over a Delaunay triangulation. We then analyze the example from the point of view of dual graphs, and reveal inconsistencies in the theory (cf. Section 6), Section 7 outlines two possible strategies to remedy the situation.

2 Voronoi diagrams for segmentation

Franz Aurenhammer has written in [2]: "Human intuition is often guided by visual perception. If one sees an underlying structure, the whole situation may be understood at a higher level." This sentence could explain the use in this technical report of the Voronoi diagram and the pyramidal approach for the segmention of grey-level images.

The definition of the Voronoi diagram is very simple [14]. Let S be a set of N points in the plane, indexed by $i \in \{1, \dots, N\}$. The *Voronoi region* associated to one point $p_i \in S$ denoted by $Vor_S(p_i)$ is the set of the points closer to p_i than to any other points of S. According to this definition it is easy to show that each Voronoi region is polygonal and convex as an intersection of the half-plane. Let us denote $H(p_i, p_j)$ the half-plane containing p_i that is defined by the perpendicular bissector of $\overline{p_i p_j}$. We can write:

$$Vor_S(p_i) = \bigcap_{i \neq j} H(p_i, p_j).$$

The *Voronoi diagram* is defined by the set of all Voronoi polygons.

An interesting property is that the dual graph of the Voronoi diagram is the *Delaunay graph* with the following properties: the Delaunay graph is a triangulation such that each circle C circumscribed by every triangle $\overline{p_i p_j, p_k}$ does not contain in its interior any point of S (Figure 1). The proof is very easy. Assume that it exists a point p_l of S in the interior of C. Then the distance between the center c of C and p_l is smaller than the distance between c and any $p_n \in S$, $n \neq l$. According to the definition of a Voronoi polygon, c belongs to the interior of $Vor_S(p_l)$ which is contradictory.

In this section we address the problem to give a description of an image with various shape: Voronoi polygons structured in the Delaunay graph. According to our goal, we apply the split and merge method [8] based on Voronoi polygons [7]. The global principle

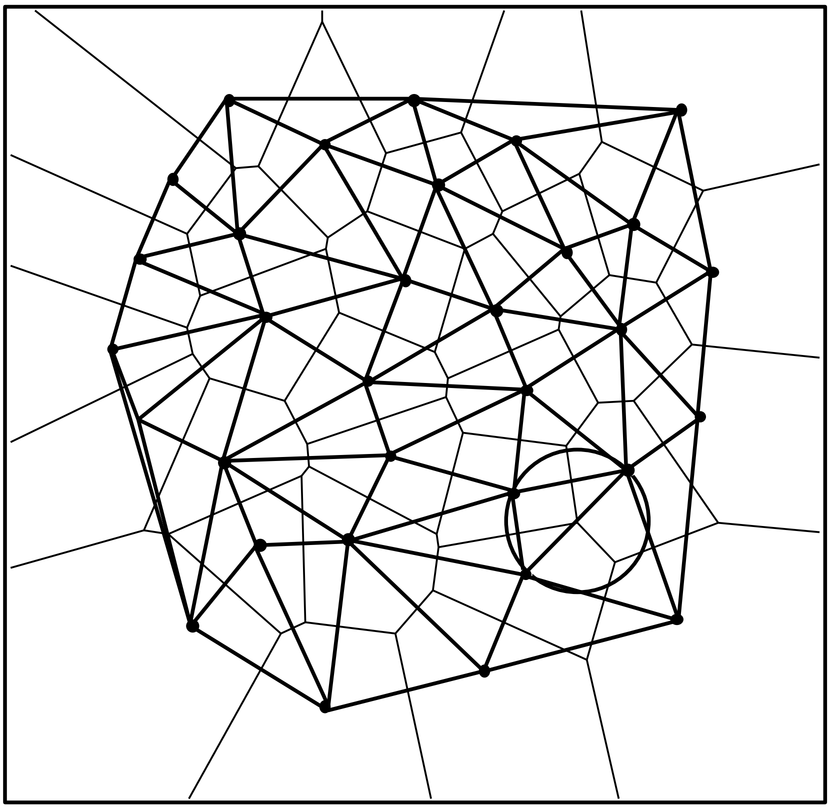


Figure 1: Voronoi polygons and Delaunay triangulation. The circle does not contain in its interior any point of S (Black dots).

results in the steps of the following algorithm (cf. Definitions 1 and 2):

1. Assign a little number of points with a Poisson process in the image.
2. Compute the Voronoi diagram and the Delaunay graph.
3. Compute mean grey value, standard deviation, and surface of each polygon.
4. For all polygons, split if the polygon is not homogeneous (Definition 1).
5. Repeat 2–4 until convergence (all the polygons are homogeneous).
6. Merge: supression of the useless polygons (Definition 2).

Definition 1 *A region enclosed by a polygon is said to be homogeneous if and only if the variance in the region is less than a given threshold.*

Definition 2 *A polygon $Vor_S(p_i)$ is said to be useless if and only if all the neighbors of $Vor_S(p_i)$ have almost equal grey level means.*

Steps 2 and 5 of our algorithm allow a dynamic management of the Voronoi and the Delaunay diagrams, illustrated in Figure 2. Here, we use the incremental technique to compute the Voronoi diagram [6] because of its two main advantages:

- the run time optimality in general cases;
- the dynamic management of the Voronoi and Delaunay structure.

The algorithm offers the possibility to add or delete one point in an existing Voronoi diagram without the need to recompute the whole structure again. Practically, we obtain

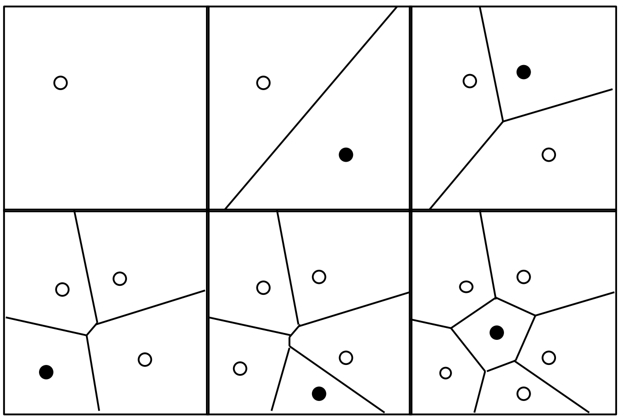


Figure 2: Construction of the Voronoi diagram by adding successively sites and local modification of the diagramm. The black dot is the last site to add.

600000 polygons with 1200000 triangles in 4 min. on a Silicon Graphics Indigo workstation. In our algorithm, the grey level variation of a given image guides the evolution and location of the polygons. Consequently the polygon distribution is adapted to the image content: there is a high density of seeds in regions with an important grey level variation and a low density of polygons where there is little grey level variation (Figure 3).

Figure 3: Result of the split and merge algorithm. From left to right and top to bottom: original image, split result (14625 polygons) and the Delaunay graph, merge result (9941 polygons) and the Delaunay graph.

To summarize we can say that:

- Voronoi diagram is very well adapted to describe a higher level content of a given image.
- Delaunay graph is usefull to describe the neighborood.

Such a description provides a higher level understanding of the image content. In [3] we have shown how to use the structure of the Delaunay graph and the information of the Voronoi polygon in a pyramidal approach. In this paper, we would like to show how to implement this in the dual irregular pyramids. Before this, let us recall some fundamental definitions of irregular pyramids.

3 Irregular pyramids

Irregular pyramids have been introduced by Meer [12] to overcome instabilities due to the rigid structure of regular pyramids [5]. Levels of irregular pyramids are general graph structures. Irregular pyramids can be used for bottom-up segmentation when the neighborhood structure of the image is represented by a graph [13]. The adaptation of the structure to the image data has been shown in [10].

We will in this section first recall some basic definitions from graph theory and then proceed with the definition of the irregular pyramid and the process to construct it.

3.1 Basic definitions from graph theory

Definition 3 *A* **graph** $G = (V, E)$ *consists of two sets: a finite set* V *of elements called* **vertices** *and a finite set* E *of elements called* **edges**. *Each edge is a binary relation between two vertices.*

We use the symbols $v_1, v_2, v_3, \ldots$ to represent the vertices and the symbols $e_1, e_2, e_3, \ldots$ to represent the edges of the graph.

Definition 4 *The vertices* v_i *and* v_j *related by an edge* e_l *are called the* **end vertices** *of* e_l. *The edge is denoted as* $e_l = (v_i, v_j)$. *An edge is said to be* **incident** *on its end vertices.*

Figure 4 shows an example of a graph. Vertices are represented by black spots ($\bullet$), edges by straight lines.

Definition 5 *Two* **vertices** *are* **adjacent** *if they are the end vertices of some edge.*

Definition 6 *The number of edges incident on a vertex* v *of a graph* $G(V, E)$ *is called its* **degree** *or its* **valency**. *It is denoted by* $d(v)$.

Definition 7 *A vertex of degree 1 is called a* **pendant vertex**. *The edge incident on a pendant vertex is called a* **pendant edge**.

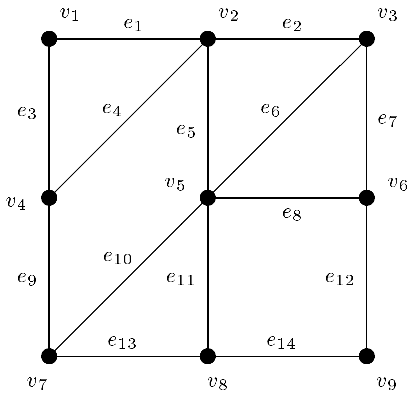


Figure 4: Pictorial representation of a graph $G(V, E)$.

3.2 Building an irregular pyramid

Definition 8 *An* **irregular pyramid** *is an ordered sequence of connected graphs $G_0, .., G_m$. The graphs of the sequence are called the* **levels** *of the irregular pyramid, G_0 the* **base level**. *Two consecutive levels G_n, G_{n+1} of an irregular pyramid are related to each other by the set inclusion $V_{n+1} \subset V_n$, and by mappings $(V_n - V_{n+1}) \longrightarrow V_{n+1}$.*

The process to construct G_{n+1} from G_n first selects set $V_{n+1} \subset V_n$ and then creates edges $e_{n+1} \in E_{n+1}$. Definition 9 contains the first version of the process given by Meer [12]. In this version, the selection of the vertex subset is based based on random numbers. For the sake of completeness we should mention that this selection has also been carried out by means of an energy term of the kind used for Hopfield neural networks [3].

Definition 9 **Stochastic decimation** *of a graph [12] is a process that is executed in the following steps:*

1. *Random numbers are assigned to the vertices.*
2. *Vertices with a local maximum of this variable* **(surviving vertices)** *are selected.*
3. *All* **non-surviving vertices** *are assigned to one survivor out of their neighborhood.*
4. *Repeat steps 1 . . . 3 for all non-survivors who do not have a survivor in their neighborhood.*
5. *The* **receptive field** *$RF(v_s)$ of a survivor v_s is formed by all non-survivors* **(children)** *that are assigned to v_s* **(parent)**. *It also includes v_s itself.*
6. *Vertices of the reduced graph are neighbors if vertices of their receptive fields are neighbors in the original graph.*

The following two locally observable rules control the extraction of survivors:

1. No two neighbors must survive.

2. Every non-survivor must have at least one survivor in its neighborhood.

All levels of an irregular pyramid are obtained from the base level G_0 of the irregular pyramid by recursive application of stochastic decimation.

4 The dual irregular pyramid

The importance of pyramidal processing schemes for image analysis is given by local connections of the processing elements. These allow to take decisions in constant time, a global decision is obtained at the apex of the pyramid after a logarithmic number of parallel processing steps.

The condition of localness, however, is not fulfilled in the irregular pyramid, since the degree of a vertex cannot be bounded during its construction [11]. The consequence is that logarithmic time complexity is lost for processes that run in irregular pyramids.

The concept of duality can remedy the situation, as we could prove in the same paper [11]. We considered the graphs that are dual to the levels of the irregular pyramid, and we proved that vertex degree in the dual graphs remains bounded throughout the pyramid.

This section contains in a first part the formal definition of a dual graph. The second part of the section is devoted to a process that has a level G_n of an irregular pyramid and its dual graph $\overline{G_n}$ as an input, as well as the result of decimation G_{n+1}. It selects surviving vertices in the dual graph and connects them according to the result of the decimation.

4.1 Paths, circuits, cutsets and dual graphs

This section assembles all definitions that we need to introduce the concept of duality between graphs.

Definition 10 *A* **path** *in a graph is a finite alternating sequence of vertices and edges* $v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ v_{k-1},\ e_k,\ v_k$ *such that*

1. *vertices v_{i-1} and v_i are the end vertices of the edge e_i, $1 \leq i \leq k$;*

2. *all edges are distinct;*

3. *all vertices are distinct.*

Vertices v_0 and v_k are called **end vertices of the path**, *and we refer to it as $v_0 - v_k$ path. The number of edges in a path is called the* **length of a path**.

Figure 5 shows two example paths of the graph G in Figure 4.

Definition 11 *A* **circuit** *is a path the end vertices of which are not distinct.*

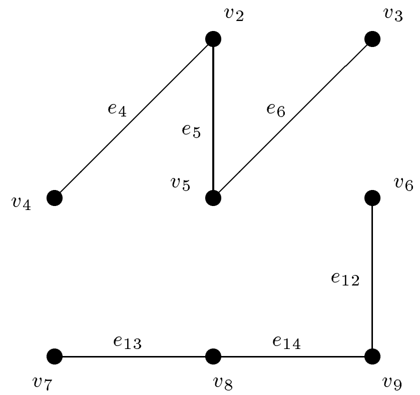


Figure 5: Two paths of G.

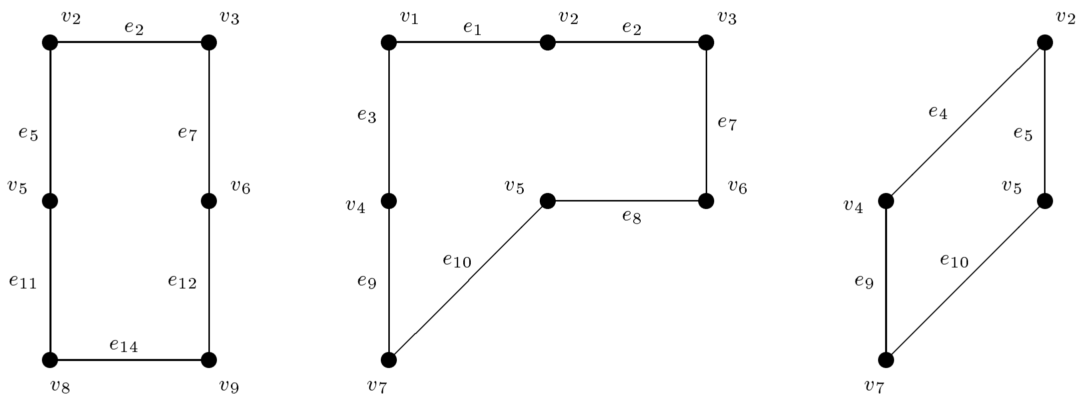


Figure 6: Three examples of circuits of G.

Figure 6 shows three circuits of the graph G in Figure 4.

Definition 12 *Let V' be a subset of the vertex set V of a graph $G = (V, E)$. Then the subgraph $G' = (V', E')$ is the induced subgraph of G on the vertex set V' (or simply* **vertex-induced subgraph** *$\langle V' \rangle$ of G) if E' is a subset of E such that an edge (v_i, v_j) is in E' if and only if v_i and v_j are in V'.*

Definition 13 *A* **cutset** *S of a connected graph $G(V, E)$ is a minimal set of edges of G such that its removal from G disconnects G, that is, the graph $G(V, E - S)$ is disconnected.*

Figure 7 shows graph G of Figure 4 after removal of cutset $\{e_9, e_{10}, e_{11}, e_{12}\}$: G is disconnected and consists of exactly two components $\langle\{v_1, v_2, v_3, v_4, v_5, v_6\}\rangle$ and $\langle\{v_7, v_8, v_9\}\rangle$.

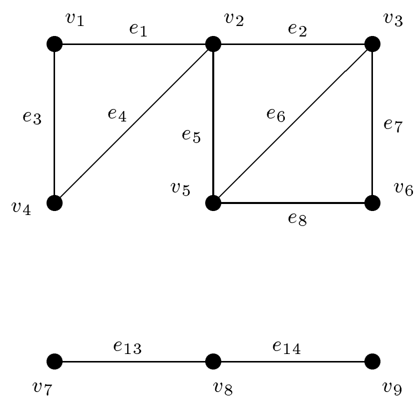


Figure 7: G after removal of cutset $\{e_9, e_{10}, e_{11}, e_{12}\}$.

Definition 14 *A graph $\overline{G}$ is a* **dual** *of a graph G if there exists a one-to-one correspondence between the edges of $\overline{G}$ and those of G such that a set of edges in $\overline{G}$ is a circuit if and only if the corresponding set of edges in G is a cutset.*

Figure 8 shows a simple example of a pair of dual graphs. Both graphs have an equal

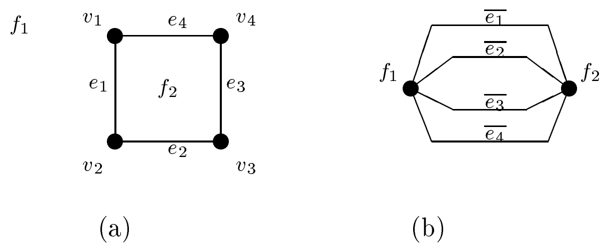


Figure 8: A pair of dual graphs.

number of edges, and there is a one-to-one correspondence between their respective circuits and cutsets. Graph G has only one circuit, $\{e_1, e_2, e_3, e_4\}$, which corresponds to the only cutset $\{\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_4}\}$ of $\overline{G}$. Table 1 lists all circuits of $\overline{G}$, along with the corresponding cutsets of G. Figure 8 also reveals another property of graphs having a dual, planarity:

Definition 15 *A graph G is said to be* **planar** *if it can be drawn on a plane surface such that its edges intersect only at their end vertices. Such a drawing of a planar graph G is called a* **planar embedding** *of G.*

circuit of $\overline{G}$	cutset of G
$\{\overline{e_1}, \overline{e_2}\}$	$\{e_1, e_2\}$
$\{\overline{e_1}, \overline{e_3}\}$	$\{e_1, e_3\}$
$\{\overline{e_1}, \overline{e_4}\}$	$\{e_1, e_4\}$
$\{\overline{e_2}, \overline{e_3}\}$	$\{e_2, e_3\}$
$\{\overline{e_2}, \overline{e_4}\}$	$\{e_2, e_4\}$
$\{\overline{e_3}, \overline{e_4}\}$	$\{e_3, e_4\}$

Table 1: Circuits of $\overline{G}$ and corresponding cutsets of G.

Definition 16 *An embedding of a planar graph on a plane divides the plane into* **regions** *or* **faces**. *A region or face is* **finite** *if the area it encloses is finite; otherwise it is* **infinite**.

Definition 17 *The edges on the boundary of a face contain exactly one circuit, and this circuit is said to enclose the face. Let $f_1, f_2, f_3, \ldots, f_r$ be the faces of a planar graph with f_r as the infinite region. We denote by $C_i, 1 \leq i \leq r$, the circuit on the boundary of region f_i. The circuits $C_1, C_2, \ldots, C_{r-1}$, corresponding to the finite regions, are called* **meshes** *or* **cycles**.

Like Harary [9, p. 103], we will refer to a cycle corresponding to a face f_i as the cycle of f_i, and we will denote it as $C(f_i)$.

4.2 Building a dual irregular pyramid

With the definition of a dual graph we can now proceed with the definition of a dual irregular pyramid.

Definition 18 *The* **dual irregular pyramid** *is an irregular pyramid, where each level is complemented by its dual graph.*

This section contains the definition of the process that complements a new level of the irregular pyramid to obtain a new level of the dual irregular pyramid. The process uses the information generated by decimation to also simplify the dual of the decimated graph. Two notions, receptive field set and valence will be used to define the process.

Definition 19 *The* **receptive field set** *$RS(f)$ of face f is the set of parents of all different receptive fields that are represented by the vertices of cycle $C(f)$ corresponding to f.*

Similarly, the receptive field set of an edge is defined.

Definition 20 *The* **receptive field set** *$RS(e)$ of edge e is the set of parents of all different receptive fields that are represented by the end vertices of e.*

Definition 21 *The* **valence** *of an edge or a face is defined as the cardinality of its receptive field set,*

$$val = card(RS).$$

Definitions 19, 20, and 21 allow to proceed with the definition of the process that complements a new level of the dual irregular pyramid.

Definition 22 *Let $\overline{G}$ and $\overline{G'}$ be the dual graphs of the graph G and the result of its decimation G' respectively. The following three steps construct a graph $G^*(F^*, E^*)$:*

1. **Induced connection set** $\overline{E_c}$*: If $val(e) = 1$ for an edge $e \in E$, the corresponding edge $\overline{e} = (f_1, f_2) \in \overline{E}$ does not create a connection in G^*. We thus define the* connection set $\overline{E_c}$ *by*
$$\overline{E_c} := \{\overline{e} \in \overline{E} | val(e) = 2\}.$$
2. **Surviving faces** F^**: Faces $f \in F$ 'survive' to F^* if their valence is greater than 2, otherwise they don't:*
$$F^* := \{f \in F | val(f) \geq 3\}.$$
3. **New edges** E^**: Two surviving faces f_0^*, f_n^* are neighbors in G^* if there exists a path $(f_0^*, f_1, \ldots, f_{n-1}, f_n^*)$ in $\overline{G_c}(F, \overline{E_c})$ and all the intermediate faces $f_1, \ldots, f_{n-1} \notin F^*$.*

Theorem 1 states the relationship between graph G^* and the dual of G', $\overline{G'}$. The proof can be found in [11].

Theorem 1 *The graph $G^*(F^*, E^*)$ constructed from $\overline{G}$ and the decimation of G according to Definition 22 is the dual of $G'(F', E')$, $G^*(F^*, E^*) \equiv \overline{G'}(F', \overline{E'})$.*

The main motivation to build the dual graph is the property of bounded degree of the graphs dual to the levels of an irregular pyramid. Theorem 2 states this property, for the proof see [11].

Theorem 2 *Let $\overline{G}(F, \overline{E})$ be the dual graph of $G(V, E)$, $G'(V', E')$ be the result of decimating $G(V, E)$, and $\overline{G'}(F', \overline{E'})$ the corresponding dual graph. For every face $f' \in F'$ there exists a face $f \in F$ such that* $\boxed{d(f') \leq d(f)}$.

5 Experimental results

Figure 9 shows six levels of a dual irregular pyramid built over a Voronoi diagram. The most interesting result of our experiments is the occurrence of pendant edges (level 4 in our example).

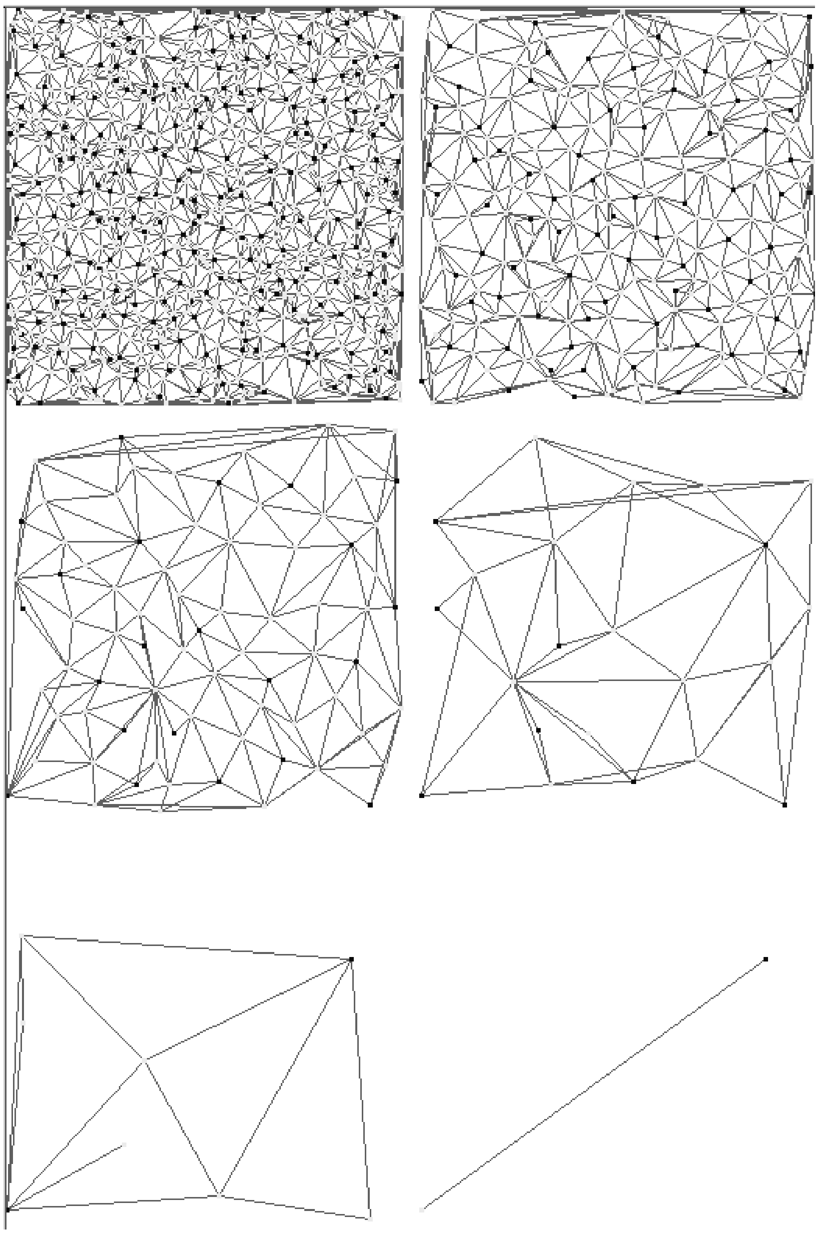
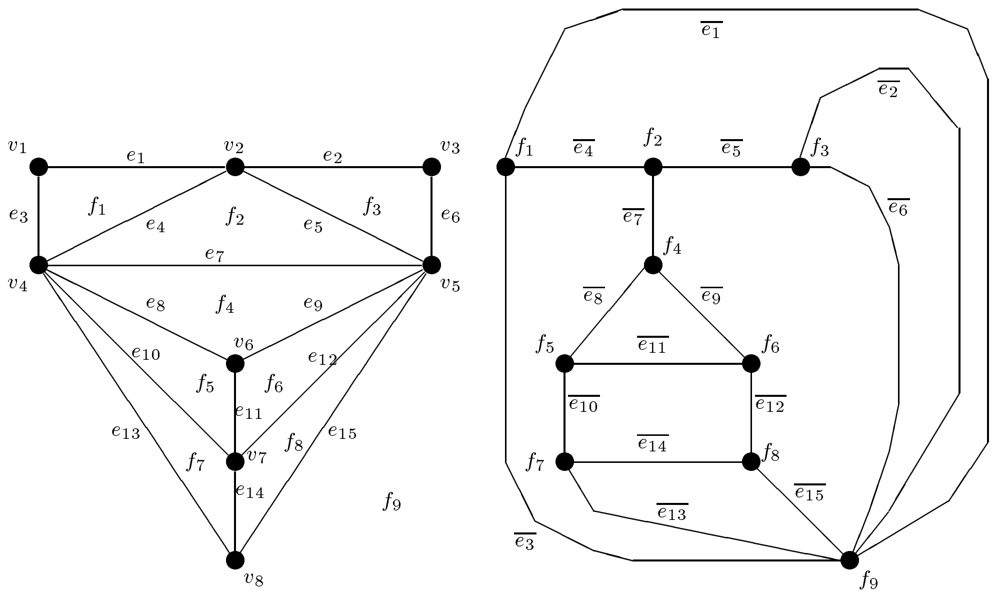


Figure 9: Dual irregular Voronoi pyramid.

6 Discussion of the results

We discuss in this section the impact of a pendant edge on the dual irregular pyramid. Figures 10, 11, 12 illustrate how decimation may lead to a pendant edge (cf. Definition 9). Figure 10 shows a graph G and its dual $\overline{G}$ before decimation of G. $G(V, E)$ consists of 8 vertices and 15 edges. The plane is divided into 9 faces, with face f_9 being the infinite region.

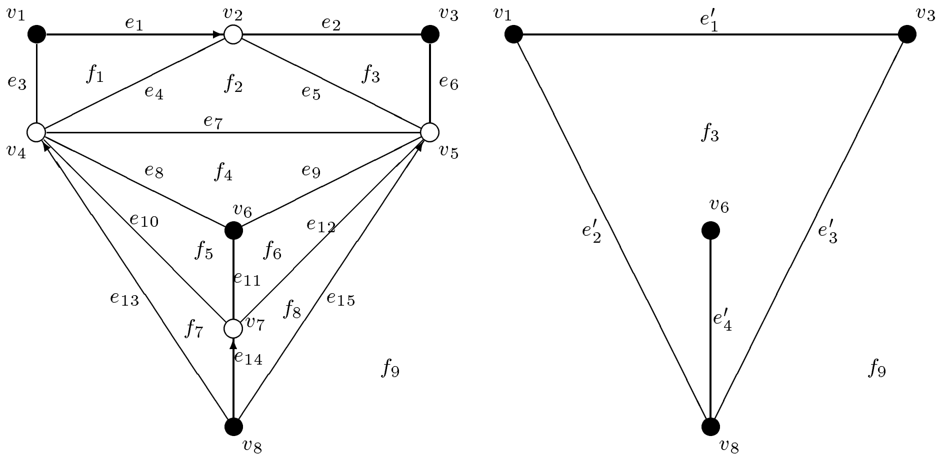
Figure 10 b shows the dual of G, $\overline{G}(\overline{V}, \overline{E})$, with one vertex representing every face of G. There is a correspondence between edges e_i of G and edges $\overline{e_i}$ of $\overline{G}$ (expressed by identical indexing), and between circuits and cutsets, respectively. Figure 11 a illustrates the result



(a) G (b) $\overline{G}$

Figure 10: G and $\overline{G}$ before decimation of G.

of the selection of survivors and the child-parent assignment. Table 2 lists the receptive fields for each vertex of the set of survivors $\{v_1, v_3, v_6, v_8\}$. Table 3 contains neighborhood relations between receptive fields, along with the neighborhood relations (edges) at the new pyramid level. Figure 11 b illustrates the result of the decimation of G. The next step during the construction of a new level of the dual irregular pyramid is to simplify also the dual of the decimated graph G. According to Definition 22, information generated by decimation is used during this step: receptive field sets and valences of faces and edges.



(a) Survivors (•), non-survivors (o), assignment (• → o)

(b) Graph G'.

Figure 11: Decimation of G, resulting graph G'.

surviving vertex	receptive field
v_1	$\{v_1, v_2\}$
v_3	$\{v_3\}$
v_6	$\{v_6\}$
v_8	$\{v_4, v_5, v_7, v_8\}$

Table 2: Survivors and receptive fields.

adjacent receptive fields	connecting edges	new edge
$RF(v_1), RF(v_3)$	e_2	e'_1
$RF(v_1), RF(v_8)$	e_3, e_4, e_5	e'_2
$RF(v_3), RF(v_8)$	e_6	e'_3
$RF(v_6), RF(v_8)$	e_8, e_9, e_{11}	e'_4

Table 3: Survivors and receptive fields.

Tables 4 and 5 contain these informations for all faces and edges. As can be seen from

face	receptive field set	valence
f_1	$\{v_1, v_8\}$	2
f_2	$\{v_1, v_8\}$	2
f_3	$\{v_1, v_3, v_8\}$	3
f_4	$\{v_6, v_8\}$	2
f_5	$\{v_6, v_8\}$	2
f_6	$\{v_6, v_8\}$	2
f_7	$\{v_8\}$	1
f_8	$\{v_8\}$	1

Table 4: Receptive field sets and valences of faces.

the valences in Table 4, only face f_3 is surviving. From the valences displayed in Table 5

edge	receptive field set	valence
e_1	$\{v_1\}$	1
e_2	$\{v_1, v_3\}$	2
e_3	$\{v_1, v_8\}$	2
e_4	$\{v_1, v_8\}$	2
e_5	$\{v_1, v_8\}$	2
e_6	$\{v_3, v_8\}$	2
e_7	$\{v_8\}$	1
e_8	$\{v_6, v_8\}$	2
e_9	$\{v_6, v_8\}$	2
e_{10}	$\{v_8\}$	1
e_{11}	$\{v_6, v_8\}$	2
e_{12}	$\{v_8\}$	1
e_{13}	$\{v_8\}$	1
e_{14}	$\{v_8\}$	1
e_{15}	$\{v_8\}$	1

Table 5: Receptive field sets and valences of edges.

follows the induced connection set $\overline{E_c}$ of edges of the dual graph:

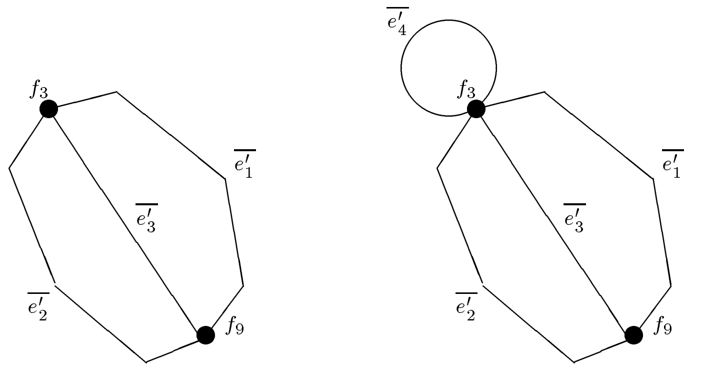
$$\overline{E_c} = \{\overline{e_2}, \overline{e_3}, \overline{e_4}, \overline{e_5}, \overline{e_6}, \overline{e_8}, \overline{e_9}, \overline{e_{11}}\}.$$

Set $\overline{E_c}$ allows to define paths in $\overline{G}$ that lead to neighborhood relations between surviving faces, in the example under consideration between f_3 and the infinite region f_9. Three paths in $\overline{G}$ can be distinguished,

- $f_3, \overline{e_5}, f_2, \overline{e_4}, f_1, \overline{e_1}, f_9$;
- $f_3, \overline{e_2}, f_9$;
- $f_3, \overline{e_6}, f_9$.

The resulting graph G^* is represented in Figure 12 a. Figure 12 b shows the dual of graph G'. It has been obtained from the planar embedding of G' in Figure 11 b by a process that is described in [15, p. 194]:

1. place a vertex in every region f_i;
2. for each edge e common to regions f_i and f_j (not necessarily distinct), draw a line connecting vertices representing f_i and f_j, so that it crosses e; this line represents the edge $\overline{e}$.



(a) Graph G^*. (b) Graph $\overline{G'}$.

Figure 12: Graph G^* and the dual graph $\overline{G'}$ are not identical.

We summarise our analysis with two observations:

- If the decimation of a graph G yields a pendant edge, then the graph G^* obtained from the process described in Definition 22 is not the dual of the decimation G'.

  This is a contradiction to Theorem 1.

- The degree of vertex f_3 in $\overline{G'}$ (Figure 12) is higher than its degree in $\overline{G}$ (Figure 10).

  This is a contradiction to Theorem 2.

7 Conclusion

Motivated by peculiar results during the construction of the dual irregular pyramid, we analyzed the algorithm carefully. We found that, starting from a triangular net, we may obtain a graph with pendant edges, i.e. the obtained topology being no longer a valid triangulation.

A more serious aspect with respect to the principle of massively parallel computation is that the duality relation between the graphs may get lost in the pyramid, and that the degree in the dual graph may also increase. Two strategies can be followed to solve the problem:

- **the conservative one:** how can we extend the formal system to cope with the above described problem ?

- **the radical one:** how else can we connect the set of surviving vertices, e.g. by triangulation ?

Further research we plan will address the following questions:

- Are there possible solutions for both strategies ?

- Which are the computational costs for the solutions ?

- Can both strategies be combined, i.e. can problem areas be identified **and** bounded, in the limits of which a radical solution can be found ?

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