Controlling Topology Preserving Graph Pyramids

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IC PyR+AmId 2022
Who recognizes the picture Coral?
Who recognizes the picture Cat?
Which one is the original image Fish?
SCIS re-produced with much less colors

Structurally Correct Image Segmentation (SCIS) by preserving the image structure/topology.
Pictures from the Berkeley Image data base [MFTM01]

<table>
<thead>
<tr>
<th>Picture</th>
<th>pixels</th>
<th>regions</th>
<th>reduction by</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coral</td>
<td>154401</td>
<td>12352</td>
<td>92%</td>
</tr>
<tr>
<td>Cat</td>
<td>154401</td>
<td>9264</td>
<td>94%</td>
</tr>
<tr>
<td>Fish</td>
<td>154401</td>
<td>9264</td>
<td>94%</td>
</tr>
</tbody>
</table>

Overview

- 4 Motivations + Background
- Building LBP-Pyramid + reconstruction
- an INSIGHT problem
- Critical Points from LBP
- Monotonic Path/Curve
- Process + Control + Tasks
- The space between critical points: slopes
- Conclusion and Outlook
1. Biological Perception Problem [Uhr86]

Vision is very parallel (= serial)

10⁷ cones, 10⁸ rods (each eye)

10³ operations/sec (1.5 msec synapse)

200 - 800 msec to recognize complex
30 msec per frame for moving object
(subtract 120 msec for periphery)

So serial depth only 25 - 600

complex vision in
< 600 serial steps
↓
parallel (MPP)
Log.complexity
↓

Leonard Uhr proposed Pyramids 1986 7/47
Leonard Uhr proposed Pyramids 1986

- "pyramid needs augmentation"
- "... any connected (data-flow) graph could be used."
- "combine bottom-up and top-down"
2. Retina is irregular

...NOT ARRAY!
⇒ graphs
⇒ irregular embeddings
3. Water’s Gateway to Heaven (2020-2024)

https://waters-gateway.boku.ac.at/

- 3D imaging and modeling of transient stomatal responses in plant leaves
- $\mu$CT images $2000 \times 2000 \times 2000$, at 2-4 times
- visible objects: different cells, waterways, airspace
- leaves are deformable...
- **understand** opening and closing of stomata for **photosynthesis**
4. Critical/stationary Points

Jan Koenderink [Koe84]: ”The Structure of Images”.

- Intensity $\Phi(x, y, t)$, $(x, y)$-coordinates, $t$ scale
- ... generated by convolution with Gaussian kernel $\Phi(x, y, 0) \ast G(t)$
- **Diffusion** $\Delta \Phi = \Phi_t$ is the basis for scale space theory.
- ”Any feature at a coarse resolution is required to possess a ’cause’ at finer resolution.”

- stationary (critical) points $\Phi_x = \Phi_y = 0$
  - Hessian $\Phi_{xx}\Phi_{yy} - \Phi_{xy}^2 \geq 0$... extremum
  - Hessian $\Phi_{xx}\Phi_{yy} - \Phi_{xy}^2 < 0$... saddle point
- **Alternative: LBP**
  - no bit switches
  - > 2 bit switches
- Extrema and saddle points disappear pairwise when $t$ increases.
**LBP-Pyramid [Cerman 2014]** CTIC Timisoara

$|V| = 5000$

Pheasant 154401 Pixels

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**Bottom-up construction:**
- each level = E-RAG $G = (V, E)$
- edge contraction: lowest contrast first
- simplify/remove multi-edges: highest contrast first
- preserve relevant critical points, determined by LBP

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$|V| = 5000$ parents

$\leq 5000$ colored regions

**Top-down reconstruction:**
- by canonical representation [TK14]
- edge de-contraction
- (removed) edge re-insertion
- instead of correcting value: children inherit parent’s value

An Insight Problem from [Piz22] 12/47
An Insight Problem from [Piz22]

Create $n$ equilateral triangles ($\triangle$) with $m$ matchsticks:

- $m=3$  $\Rightarrow$  $1 \triangle$:

- $m=5$  $\Rightarrow$  $2 \triangle$:

6 Matchsticks give 4 Triangles? 13/47
6 Matchsticks give 4 Triangles?

$m=6 \implies 4 \triangle$

Euler-Poincaré characteristic

\[
\#P - \#E + \#F = 1
\]

\[
\bullet - m + \triangle =
\]

\[
3 - 3 + 1 = 1
\]

\[
4 - 5 + 2 = 1
\]

\[
? - 6 + 4 = 1
\]

Insight Problem = difficult
6 Matchsticks give 4 Triangles

\[ m = 6 \implies 4 \triangle: \]

Euler-Poincaré in 3D

\[
\#P - \#E + \#F - \#V = 1
\]

\[
\bullet - m + \triangle - = \\
4 - 6 + 4 - 1 = 1
\]

1. Solution to insight problem: \textbf{Change representation}, 2D \rightarrow 3D
2. ”\textbf{AHA}”!!! ...easy to explain.
3. solvable by \textbf{optimization? learning?}

Let’s consider LBPs in 1D...
Critical Points of a height Profile

- Critical points in 1D are local extrema
- ... characterized by horizontal tangents
- curves between critical points are monotonically
- ... alternating ups (↑) and downs (↓)

LBPs along monotonic curves 16/47
Local Binary Patterns (LBP) compare a central point with its neighbors,
0 ... neighbor is smaller
1 ... neighbor is greater

along monotonic curves: same (‘uniform’) LBP-code 0 · 1
bounded by local minimum (⊖): 1 · 1
and local maximum (⊕): 0 · 0

no derivatives!
Monotonic Curves/Paths $\pi$

1D curve: $G(V,E)$

- edge/curve between $\oplus \longrightarrow \ominus$ is monotonically decreasing.
  
  ... not necessarily smooth!

- Monotonic path $\pi(p_1,p_n) = (p_1, \ldots , p_n)$:
  
  $(g(p_{i+1}) - g(p_i))\sigma \leq 0 \quad \forall i \in [1,n-1]$ and given $\sigma \in \{-1, +1\}$.

- LBP bits $\longleftrightarrow$ oriented edge
  
  $\implies$ allows LBP for vertices with different degrees

- CONTRACT edge with lowest contrast
  
  $\implies$ preserves monotonicity, shortens $\pi$, preserves critical points, ...
Critical Points in 2D ...

can be recognized also by (LBP) Ojala, Pietikäinen, Harwood [OPH96]

Orientation of edges:

\[ \begin{align*}
\oplus & \text{ local maximum} : \\
\ominus & \text{ local minimum} \\
\otimes & \text{ local saddle}
\end{align*} \]

downwards

To preserve in the hierarchy of the primal graph: \( \oplus, \ominus, \otimes \)
Lowest contrast pairs \((\oplus, \ominus), (\oplus, \otimes), (\otimes, \ominus)\) can be dropped (as by [Koe84]).

\[ \Rightarrow \text{ Contraction Kernels} \]
Contraction Kernels

... connect 10 NON-Survivors with 6 Survivors
Edge Contraction

... shrinks 10 selected edges into 6 Survivors
Edge Contraction

... further shrinks selected edges into Survivors
Contracted Graph

... until they disappear, multiple edges may appear
Simplification: remove multiples?

Consider degree of 9 faces (=dual vertices) $A, \ldots, I$
Dual Operations: Contraction vs. Removal

merge degree 2 faces $A, C, F, G$ = remove 4 double edges
Simplify: Remove multi-edges = contract duals

...shrinks 4 edges into dual vertices $B, D, E$ with degree $> 2$
After 1st Simplify: 1 double edge bounds $I$

Repeat simplification by merging faces $I$ and $E$
Dually Contracted Graph

All remaining 4 faces $B, D, E, H$ have degree $>2$
BOTTOM-UP CONSTRUCTION

while further abstraction is possible do

1. select contraction kernels
2. perform contraction
3. and simplification;
4. apply reduction functions $\rightarrow$ new reduced content

Each iteration: new level of the pyramid
Preserving Topology

Changes $\Delta$ by the primitive operations, edge contraction and edge removal:

<table>
<thead>
<tr>
<th>operation</th>
<th>$\Delta #P - \Delta #E + \Delta #F = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>contraction</td>
<td>$1 - 1 + 0 = 0$</td>
</tr>
<tr>
<td>removal</td>
<td>$0 - 1 + 1 = 0$</td>
</tr>
</tbody>
</table>

Any number of contractions and removals does NOT change the characteristic!
CONTROL by the CONTENT

1. Selection of contraction kernels
2. Simplification strategies
3. Reduction functions
4. Expansion
   (a) by inverse operations (as in [TK14]):
      | de-contraction | contraction |
      | re-insertion   | removal     |
   (b) by interpolation
   (c) by inheritance
   (d) by model refinement
1. SELECT Contraction KERNELs

- RANDOM (like stoch.pyramid $\rightarrow$ [Mee89])
- Filters $\rightarrow$ local MAX., the ADAPTIVE Pyramid [JM92]
- RULES (CCL, GAP closing, line drawings [BK99, BK21])
- MATCHING: parametric MODELS (e.g. by correlation) 'goodness of match' structures (e.g. by GRAPH MATCHING)
- locally lowest contrast, critical points survive
2. Simplification Strategies

- complete simplification after each contraction
  \( \mathcal{O}(a^{-1}(n, n)) \enspace a(n, n) \ldots \text{Ackermann function} \)
- only one simplification pass after a contraction
- ALL simplifications after ALL contractions
- content-controlled simplification (some degree 1 or 2 faces survive)
- anticipated simplification before contraction [BBK22]
3. REDUCTION FUNCTIONS

- Inheritance (e.g. for CCL)
- average, convolution filter, \textbf{as in DCNN}
- transitive closure (used in line drawings [BK99])
- MODEL name, symbol (like \(\bigcirc\), \(\bullet\), \(\ominus\) for dotted lines in [Kro95])
- parameters that best match data [HR84].
- LBP: survivors inherit value of critical point \textbf{preserves range of grey values}
## Selecting Parameters for Abstraction (1)

<table>
<thead>
<tr>
<th>Application</th>
<th>Important elements survive</th>
<th>Negligible are merged</th>
<th>Redundant are removed</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCL [KM95, MK95, Kro96, BK21]</td>
<td>1 repr/lab</td>
<td>(L, L)</td>
<td>empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>segmentation [KB96b, KB96a, KH04a, HMK04]</td>
<td>1 repr/ region</td>
<td>similar, end points</td>
<td>empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>2x on curve [Kro97]</td>
<td>X, ends</td>
<td>empty space, connections</td>
<td>empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>line images [KBI98, BK98, KB98]</td>
<td>ends, junctions</td>
<td>empty space, connections</td>
<td>empty faces, no touching curve</td>
</tr>
</tbody>
</table>
## Selecting Parameters for Abstraction (2)

<table>
<thead>
<tr>
<th>Application</th>
<th>Important</th>
<th>Negligible</th>
<th>Redundant</th>
</tr>
</thead>
<tbody>
<tr>
<td>matching</td>
<td>[PKJ98, GPK02, GPK04] discrim.template, object boundary</td>
<td>simil.inside object</td>
<td>simil.empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>motion</td>
<td>[GEK99, MKH04, AIK09] foreground, static background, articulations</td>
<td>occluded backgr. moving foregr.</td>
<td>empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>gap closing</td>
<td>[Kro02, KH04b] 1 repr/lab incl. background</td>
<td>(L, L)</td>
<td>empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>RAG + Hierarchy</td>
<td>[HK03, HK04, HIK06] max.ext.Contrast, MST</td>
<td>min.int.Contrast</td>
<td>empty faces, deg &lt; 3</td>
</tr>
<tr>
<td>LBP-Pyramid</td>
<td>[CGK15, CJGK16, KCBG19a] critical points, texture, high freq.</td>
<td>lowest contrast</td>
<td>empty faces, deg &lt; 3</td>
</tr>
</tbody>
</table>
Original Berkely Pheasant 154401 Pixels
SCIS(Pheasant) with 30\%(154401) Regions
SCIS(Pheasant) with 10\%(154401) Regions
SCIS(Pheasant) with 3%(154401) Regions
’Image = Structure + Few Colors’, [BGK21]

- most critical points survive
- contracting lowest contrast \(\longleftrightarrow\) preserving high contrast of dual edge
- In contrast to ALL smoothing reductions:
  - preserves high frequencies (small, thin details)
- \(\Rightarrow\) reconstructions with only a few highest levels give good results

...what are the spaces between critical points?
'On the Space Between Critical Points', [KCBG19b]

A connected region $R$ of a 2D surface is a **slope region** iff all pairs of points $\in R$ are connected by a **continuous monotonic curve** $\in R$.

- there may be one $\oplus, \ominus$ inside $R$
- saddles $\otimes$ have an important role: $\otimes$ **are only on the boundary** $\partial R$, never inside $R$
- more ................................................................. [BHK19]
Level Curves

mesh of slope region

bounded by level curve

level curves of slope region. Extrema $\rightarrow$ closed level curves. Hole bounded by level curve.
An abstract cellular complex is a **slope complex** if all cells are slope regions.

Bounding relations are given by $G(V, E) \leftrightarrow \overline{G}(\overline{V}, \overline{E})$

More ... [GDBCK21]
Properties of Topological Pyramids

1. Graphs need **multi-edge and self-loop**, are non-simple!
   \( \{ \oplus, \ominus, \otimes \} \)

2. dual graph pyramid
   \[ \text{primal graph } G(V, E) \leftrightarrow \text{dual graph } \overline{G}(\overline{V}, \overline{E}) \]

3. controlled by application specific
   - selection criteria,
   - reduction functions (could be convolution + activation), and
   - termination criteria

4. pseudo elements characterize topological relations
   2D-ex.: pseudo edge \( \leftrightarrow \) hole, 3D-ex.: pseudo face \( \leftrightarrow \) tunnel, \ldots

5. independence of operations: parallelism see [BBK22]@ICPRAI
Conclusion (1)

• intrinsic 3D (cell-)structure does not change at higher levels
• topological data structures:
  – 2D: planar graphs, combinatorial maps
  – 3D: combinatorial maps, generalized maps
  – nD: generalized maps.
• hierarchies (pyramids) can be built on image, retina, planar graph, combinatorial map, generalized map
• Goal: REDUCE DATA while PRESERVING PROPERTIES
Conclusion (2) and Outlook

- LBP $\leftrightarrow$ edge orientation $\rightarrow$ critical points $\oplus, \ominus, \otimes$ and slopes.
- Partitioning into slope regions: not unique
- Slope complex partitions continuous surface.
- Merge slopes $\Rightarrow$ dual graph pyramid.
- Any hill-climbing inside a slope region reaches the peak!
- Complexity: $O(\log(\text{diameter (slope)}))$ following parent links.
- DCNN $\leftrightarrow$ ??
## Contributions by ...

<table>
<thead>
<tr>
<th>1990</th>
<th>2000</th>
<th>2020</th>
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</thead>
<tbody>
<tr>
<td>Annick Montanvert</td>
<td>Adrian Ion</td>
<td>Nicole Artner</td>
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<tr>
<td>Peter Meer</td>
<td>Yll Haxhimusa</td>
<td>Luís Mateos</td>
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<tr>
<td>Cornelia Fermüller</td>
<td>Luc Brun</td>
<td>Ines Janusch</td>
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<td>Pedro Real Jurado</td>
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<td>Rocio Gonzalez-Diaz</td>
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<td>Rocío M. Casablanca</td>
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<tr>
<td>Dieter Willersinn</td>
<td>Mabel Iglesias-Ham</td>
<td>Martin Cerman</td>
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<td>Jean-Michel Jolion</td>
<td>Dan Shao</td>
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<td>Jean-Gerard Pailloncy</td>
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<td>Mark Burge</td>
<td>Samuel Peltier</td>
<td>Majid Banaeyan</td>
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<tr>
<td>Luc Brun</td>
<td>Pascal Lienhardt</td>
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<tr>
<td></td>
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<td>Jiti Hladuvka</td>
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are cordially acknowledged. New future collaborations welcome!
References


