Technical Report

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Pyramids with Combinatorial Maps

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Abstract

This paper presents a new formalism for irregular pyramids based on combinatorial maps. This technical report continue the work began with the TR-54 report [16]. Definition and properties of Contraction kernels are generalized and completed. The definition and properties of Equivalent contraction kernels are also given.

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1 Introduction

The multi-level representation of an image called pyramid [8, 15] allows us to define different levels of representation of a same object. This method introduced by Pavlidis [8] defines several partitions of a same image and link each connected components defined at one level with its decomposition in the next level. The top of a pyramid, is usually composed of only one connected region describing the whole image while its base describes the lowest level of representation available on the image. For example, given a grey-scale image, the base of a pyramid can be composed of connected components having the same grey level. Another usual way to define the base of the pyramid consists to define each pixel of the input image as a basic region.

Recently, graphs have been used more frequently for representing and processing digital images. Typically such graphs represent the pixel neighborhood, the region adjacency, or the semantical context of image objects. In analogy to regular image pyramids, dual graph contraction [10] has been used to build irregular graph pyramids with the aim to preserve the high efficiency of the regular ancestors while gaining further flexibility to adapt their structure to the data. Experiences with connected component analysis [14], with universal segmentation [12], and with topological analysis of line drawings [11, 13] show the great potential of this concept.

In the present document, we study the definition and the properties of graph-pyramids defined by Combinatorial maps [6]. Basic definitions and properties of Combinatorial maps used in this document may be found in a previous technical report [16]. Moreover, some definitions given in [16] are generalized and completed in this document.

The rest of the paper is organized as follow: In section 2 we define the contraction kernel notion in term of combinatorial maps. In section 3 we study the successive application of several contraction kernels.

2 Contraction Kernel

We will provide in this section a definition of a tree and a forest. These definitions will be used to define a contraction Kernel and the **connecting walk map** deduced from it. Finally we will show that the connecting walk map is isomorph to a given contracted map.

2.1 Partition and Disjoint Vertex set

Definition 1 Vertex Partition

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, $\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n \subset \boldsymbol{\mathcal{D}}$ is a vertex-partition of G iff:

1. *All $\boldsymbol{\mathcal{D}}_i$ are non-empty:*

$$\forall i \in \{1, \ldots, n\} \quad \boldsymbol{\mathcal{D}}_i \neq \emptyset$$

2. *Each set $\boldsymbol{\mathcal{D}}_i$ is symmetric:*

$$\forall i \in \{1, \ldots, n\} \quad \alpha^*(\boldsymbol{\mathcal{D}}_i) = \boldsymbol{\mathcal{D}}_i$$

3. *Each vertex may be retrieved thanks to a dart in one $\boldsymbol{\mathcal{D}}_i$:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \exists i \in \{1, \ldots, n\}, \quad \exists d' \in \boldsymbol{\mathcal{D}}_i \quad | \quad d \in \sigma^*(d')$$

4. *The set of darts of one vertex is included in only one $\boldsymbol{\mathcal{D}}_i$:*

$$\forall i, k \in \{1, \ldots, n\}^2, i \neq k \quad \sigma^*(\boldsymbol{\mathcal{D}}_i) \cap \sigma^*(\boldsymbol{\mathcal{D}}_k) = \emptyset$$

This last definition generalizes the one given in [16] in order to fit with the usual notion of a vertex partition (see Figure 1(b)).

Definition 2 Disjoint Vertex Set

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, $\boldsymbol{\mathcal{D}}_1, \ldots, \boldsymbol{\mathcal{D}}_n \subset \boldsymbol{\mathcal{D}}$ is a disjoint vertex-set of G iff:

1. *All $\boldsymbol{\mathcal{D}}_i$ are non-empty:*

$$\forall i \in \{1, \ldots, n\} \quad \boldsymbol{\mathcal{D}}_i \neq \emptyset$$

2. *Each set $\boldsymbol{\mathcal{D}}_i$ is symmetric:*

$$\forall i \in \{1, \ldots, n\} \quad \alpha^*(\boldsymbol{\mathcal{D}}_i) = \boldsymbol{\mathcal{D}}_i$$

3. *All darts of one vertex belong to exactly one $\boldsymbol{\mathcal{D}}_i$:*

$$\forall i, k \in \{1, \ldots, n\}^2, i \neq k \quad \sigma^*(\boldsymbol{\mathcal{D}}_i) \cap \sigma^*(\boldsymbol{\mathcal{D}}_k) = \emptyset$$

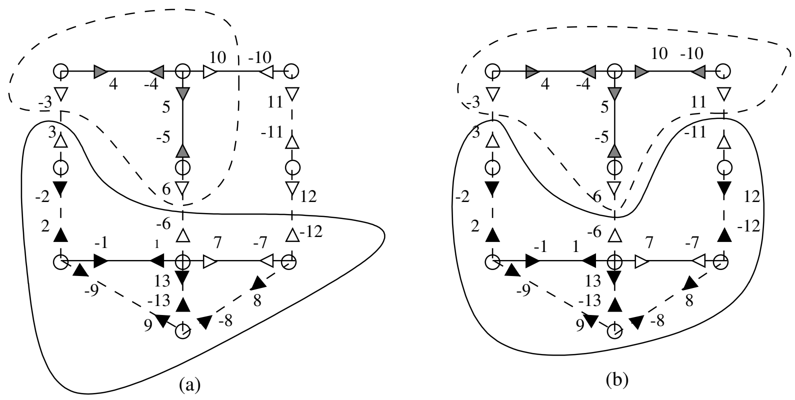

Figure 1: *Example for (a) disjoint vertex set and (b) vertex-partition*

This definition relaxes condition (3) of a vertex-partition (definition 1) in order to allow some vertices to be unaffected by the operations which may be performed on the disjoint vertex-set (see Figure 1(a)).

Proposition 1 *Given a combinatorial map* $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$*, and a dart* d*. The sub combinatorial map:* $G' = (\alpha^*(\varphi^*(d)), \sigma', \alpha)$ *isolates the face* $\varphi^*(d)$ *of* G*. The finite face in G' is bounded by the same darts*

$$\varphi'^*(d) = \varphi^*(d)$$

Proof:

If $d' \in \varphi^*(d)$ we have:

$$\varphi'(d') = \sigma^n(\alpha(d')) \quad \text{with} \quad n = Min\{p \in \mathbb{N}^* \quad | \quad \sigma^p(\alpha(d')) \in \alpha^*(\varphi^*(d))\}$$

Since $\sigma(\alpha(d')) = \varphi(d') \in \alpha^*(\varphi^*(d))$ we have, $n = 1$ and $\varphi'(d') = \varphi(d')$.

Let us denote the two permutations:

$$\begin{array}{rcl} \varphi'^*(d) & = & (d'_0 = d, d'_1, \ldots, d'_r) \\ \varphi^*(d) & = & (d_0 = d, d_1, \ldots, d_s) \end{array}$$

Let q denote the length of the shorter permutation $Min(r, s)$, and let us make the following recurrence hypothesis on k:

$$\forall i \in \{0, \ldots, k\} \quad d_i = d'_i$$

The hypothesis is true for $k = 0$, let us suppose that it remains true for a given $k > 0$. Then $d'_{k+1} = \varphi'(d'_k) = \varphi'(d_k) = \varphi(d_k) = d_{k+1}$. Thus the property holds until $k + 1$. Moreover, due to this property, we must have $r = s$. Indeed, if $r < s$ we have:

$$d = \varphi'(d'_n) = \varphi(d_n) = d_{n+1}$$

And in this case φ is not a permutation since a same dart can appear at most once in one orbit. □

2.2 Tree and forest

Two adjacent regions of a partition merge if the separating boundary segment is removed. The resulting larger region can merge with any of the new neighbors and so forth. Each time one of the separating boundary segments is removed. Boundary segments are encoded by darts of the dual combinatorial map $\overline{G}$. Since any removal in $\overline{G}$ corresponds to a contraction in G the sequence of removals of boundary segments corresponds to a sequence of contractions in G. Since self-loops cannot be contracted a sequence of successive contractions may not contain a circuit (see [16]). Thus the set of darts involved must form a tree (see definition 3 below) or a forest (see definition 4 below).

Definition 3 Map tree

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a set $\boldsymbol{\mathcal{D}}$' will be called a subtree of G iff $\alpha^*(\boldsymbol{\mathcal{D}}') = \boldsymbol{\mathcal{D}}'$ and the submap:

$$G_T = (\boldsymbol{\mathcal{D}}', \sigma' = \sigma \circ p_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{D}}'}, \alpha)$$

is connected and has only one φ'-orbit.

The Tree definition will be used to contract a set of vertices into a single vertex. More generally, if we contract a set of vertices into a given set of surviving vertices, the set of darts involved in such contractions may be encoded by a forest (see definition 4).

Definition 4 Forest

Given a combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, the set $\boldsymbol{\mathcal{D}}'$ will be called a forest of G iff $\alpha^*(\boldsymbol{\mathcal{D}}') = \boldsymbol{\mathcal{D}}'$ and each of its connected components $G_i = (\boldsymbol{\mathcal{D}}_i, \sigma_i, \alpha)$ is a tree.

Theorem 1 *Any sub combinatorial map of a forest is a forest.*

Proof:

See Tutte [18] □

Proposition 2 *Let $G = (\mathcal{D}, \sigma, \alpha)$, and $F \subset \mathcal{D}$ be a non-empty forest of G. If $\mathcal{CC}(F)$ denotes the set of connected components of F then each component is a tree and $\mathcal{CC}(F)$ is a disjoint vertex-set of G.*

Proof:

Each $\mathcal{T} \in \mathcal{CC}(F)$ is a forest, as a sub-combinatorial map of a forest, and connected. It is thus a tree. Moreover, since F is supposed to be non-empty, each $\mathcal{T}$ is non-empty. Let us suppose that:

$$\exists d \in \mathcal{D}, \exists(\mathcal{T}, \mathcal{T}') \in \mathcal{CC}(F)^2, \mathcal{T} \neq \mathcal{T}' \quad | \quad d \in \sigma^*(\mathcal{T}) \cap \sigma^*(\mathcal{T}')$$

and let us consider two darts $(d_1, d_2) \in \mathcal{T} \times \mathcal{T}'$. Since $\mathcal{T}$ and $\mathcal{T}'$ are connected we can find two paths P_1 and P_2 (see [16]), respectively included in $\mathcal{T}$ and $\mathcal{T}'$, which connect $\sigma^*(d_1)$ to $\sigma^*(d)$ and $\sigma^*(d)$ to $\sigma^*(d_2)$. The path $P_1.P_2$ connect $\sigma^*(d_1)$ to $\sigma^*(d_2)$ and is included in $\mathcal{T} \cup \mathcal{T}'$ which is thus connected. This is in contradiction with our definition of the set $\mathcal{CC}(F)$. Thus:

$$\sigma^*(\mathcal{T}) \cap \sigma^*(\mathcal{T}') = \emptyset$$

Therefore, $\mathcal{CC}(F)$ is a disjoint-vertex set of G. □

The notions of tree and forest are closely linked to the notion of connectivity. In particular, a unique path (see definition 7) connects two vertices of a tree. Moreover, a forest does not have any cycle (see definition 8). The paths and the cycles may be considered as particular case of a more general object called a walk.

Definition 5 Walk

Given a connected combinatorial map $G = (\mathcal{D}, \sigma, \alpha)$, a walk in G is a sequence of darts $(d_1, \ldots, d_n)$ such that:

$$\forall i \in \{1, \ldots, n-1\} \quad \alpha(d_i) \in \sigma^*(d_{i+1})$$

The walk is said to be closed if $\alpha(d_n) \in \sigma^*(d_1)$ and open otherwise [7].

According to Harary [7], different kind of walks may be distinguish:

Definition 6 Trail
A trail is a walk $W = (d_1, \dots, d_n)$ where all the edges are distinct:

$$\forall (i,j) \in \{1, \dots, n\}^2, i \neq j \quad d_i \notin \alpha^*(d_j)$$

Definition 7 Path
A walk $W = d_1, \dots, d_n$ will be called a Path if all its vertices (and thus all its edges) are distinct:

$$\begin{cases} \forall i \in \{2, \dots, n\} & \sigma^*(d_i) \cap \alpha^*(W) = \{d_i, \alpha(d_{i-1})\} \\ & \sigma^*(d_1) \cap \alpha^*(W) = \{d_1\} \end{cases}$$

Note that according to our definition a Path must be an open walk. A closed path will be called a cycle.

Definition 8 Cycle
A walk $W = d_1, \dots, d_n$ will be called a cycle if all its vertices except the first and the last one are distinct:

$$\begin{cases} \forall i \in \{2, \dots, n\} & \sigma^*(d_i) \cap \alpha^*(W) = \{d_i, \alpha(d_{i-1})\} \\ & \sigma^*(d_1) \cap \alpha^*(W) = \{d_1, \alpha(d_n)\} \end{cases}$$

The notions of paths and cycles are connected to the notion of tree by the following theorem:

Theorem 2 *The following statements are equivalent for a combinatorial map G:*

1. *G is a tree*
2. *G is connected and $p = q + 1$, where p denotes the number of vertices and q the number of edges.*
3. *G is acyclic and $p = q + 1$*
4. *Every two vertices of G are joined by a unique path*

Proof:

Equivalence between statements (2) to (4) is demonstrated in Harary's Book [7]. We have thus only to show that our definition of a tree is equivalent to the one of Harary.

Let us show that, (1) implies (2). Using our definition a tree must be connected and have only one face. Using Euler relationship we have: $p - q + 1 = 2$, therefore $p = q + 1$. Conversely, if G is connected the relationship $p = q + 1$ implies that the number of faces is equal to one. □

2.3 Contraction Kernel

In the following we will focus on connected combinatorial maps. If the combinatorial map is not connected the following definitions and propositions may be applied to every connected component of the combinatorial map. Moreover, since the vertices of a combinatorial map are implicitly defined by the darts which belong to their σ-orbits, we must require that at least on dart survives. In this last case the resulting graph is reduced to one vertex with a self loop. The two previous restriction are used in definition 9 to define a contraction kernel and the set of surviving darts.

Definition 9 Contraction Kernel

Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, the set K will be called a contraction kernel iff:

1. *K is a forest of G,*
2. *K does not include all darts of G:*

$$\mathcal{SD} = \boldsymbol{\mathcal{D}} - K \neq \emptyset$$

The set $\mathcal{SD}$ is called the set of surviving darts.

Note that, using proposition 2, if a set K of darts is a forest of the combinatorial map G, its set of connected components $\mathcal{CC}(K)$ is a disjoint vertex set. Moreover, each element $\mathcal{T}$ of $\mathcal{CC}(K)$ is a tree (see proposition 2).

The following lemma shows that a tree $\mathcal{T}$contains at least one vertex with surviving darts. This property may be understand as follow: Since the trees $\mathcal{CC}(K)$ form a disjoint vertex set, the vertices of the trees should not be directly adjacent (see the last requirement of definition 2). The combinatorial map being connected, the connection between these trees must be realized by surviving darts. Moreover, if the contraction kernel contains only one tree, some surviving darts must remains by definition of a contraction kernel. Therefore, the σ-orbit of the tree must contains these surviving darts.

Lemma 1 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a contraction kernel K, every connected component $\mathcal{T}$ of $\mathcal{CC}(K)$ has at least one vertex with a surviving dart:*

$$\forall \mathcal{T} \in \mathcal{CC}(K) \quad \sigma^*(\mathcal{T}) \cap \mathcal{SD} \neq \emptyset$$

These surviving darts connect the trees of K.

Proof:

Let us consider $\mathcal{T} \in \mathcal{CC}(K)$, a dart $d \in \mathcal{T}$ and a dart $d' \in \mathcal{SD}$. The combinatorial map G being connected we have a path $P = d_1, \dots, d_n$ from $\sigma^*(d)$ to $\sigma^*(d')$. Now let us consider the last dart in the sequence $d_1, \dots, d_n$ which belongs to $\sigma^*(T)$. Its index i is equal to:

$$i = Max\{j \in \{1, \dots, n\} \mid \forall k \in \{1, \dots, j\} \quad d_k \in \sigma^*(\mathcal{T})\}$$

Note that the index i is at least equal to 1 since $d_1 \in \sigma^*(d) \subset \sigma^*(\mathcal{T})$.

- Let us suppose that $i < n$

  Using item 3 of definition 2 we have $\sigma^*(d_i) \cap (K - \mathcal{T}) = \emptyset$. Thus $d_i \in \mathcal{T}$ or $d_i \in \mathcal{SD}$. If d_i belongs to $\mathcal{T}$, the tree $\mathcal{T}$ being symmetric we have $\alpha(d_i) \in \mathcal{T}$. Thus $d_{i+1} \in \sigma^*(\alpha(d_i)) \subset \sigma^*(\mathcal{T})$ which is in contradiction with the definition of i, thus we have $d_i \in \mathcal{SD}$. Since $d_i \in \sigma^*(\mathcal{T})$ the lemma is demonstrated.

- If $i = n$

  Then we can show, with the same kind of demonstration, that $d_n \in \mathcal{SD} \cap \sigma^*(\mathcal{T})$ or $d_n \in \mathcal{T}$. In this last case we have $\alpha(d_n) \in \mathcal{T}$ and $d' \in \sigma^*(\alpha(d_n)) \cap \mathcal{SD} \subset \sigma^*(\mathcal{T}) \cap \mathcal{SD}$.

□

Proposition 3 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a contraction kernel K not all darts of a face may disappear:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \varphi^*(d) \cap \mathcal{SD} \neq \emptyset$$

Proof:

The proposition is trivial if $d \in \mathcal{SD}$. Let us suppose that d belongs to a given $\mathcal{T} \in \mathcal{CC}(K)$. The tree $\mathcal{T}$ being symmetric we have $\alpha(d) \in \mathcal{T}$. Using item 3 of definition 2 we have $\sigma^*(\alpha(d)) \cap (K - \mathcal{T}) = \emptyset$. Thus $\sigma(\alpha(d)) \in \mathcal{T}$ or $\sigma(\alpha(d)) \in \mathcal{SD}$. Written in terms of permutation φ we obtain:

$$\varphi(d) \in \mathcal{T} \text{ or } \varphi(d) \in \mathcal{SD}$$

We can deduce from the above formula, that $\varphi^*(d)$ intersect $\mathcal{SD}$ or is included in $\mathcal{T}$.

Let us suppose that $\varphi^*(d) \subset \mathcal{T}$: we have $\varphi^*(d) = \mathcal{T}$ since the tree $\mathcal{T}$ has only one φ-orbit. Using Lemma 1 we have:

$$\sigma^*(\varphi^*(d)) \cap \mathcal{SD} = \sigma^*(\mathcal{T}) \cap \mathcal{SD} \neq \emptyset$$

We can thus consider d' in $\varphi^*(d)$ such that $\sigma(d') \in \mathcal{SD}$. Then $\alpha(d') \in \mathcal{T} = \varphi^*(d)$ and $\varphi(\alpha(d')) = \sigma(d') \in \mathcal{SD}$. Thus $\varphi^*(d) \cap \mathcal{SD} = \varphi^*(\alpha(d')) \cap \mathcal{SD} \neq \emptyset$. This is forbidden by our hypothesis $\varphi^*(d) \subset \mathcal{T}$.

□

Lemma 1 and proposition 3 will be used in the following demonstrations. However, as an immediate consequence of proposition 3, we can state that a combinatorial map with at least two faces can't be reduced to a single loop by contraction operations solely. Indeed, since one dart must survive in each face the reduced combinatorial map should have at least one self loop for each face of the initial combinatorial map. Thus the reduction of the initial combinatorial map must use contraction and dual contraction operations.

2.4 Map of Connecting Walks

In this section we define the notion of connecting walk. This notion may be considered as the extension of the definition of connecting paths defined within the Decimation Parameter framework [9, 16, 5].

Then we define an involution α_K and a permutation σ_K on the set of connecting walks. The two permutations α_K and σ_K define a combinatorial map on the set of connecting walks. We will show in the next section that this combinatorial map is isomorphic [16] to the one deduced from the contractions defined by the contraction kernel. We also study some properties of the permutations α_K and σ_K. Equivalent properties in the contracted combinatorial map may be deduced thanks to the isomorphism.

Definition 10 Connecting walk

Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, a contraction kernel K and a dart $d \in \mathcal{SD}$, the **connecting walk** *associated to d is equal to:*

$$CW(d) = d, \varphi(d), \dots, \varphi^{n-1}(d) \textit{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\}$$

Note that the connecting walk is defined for all darts d in $\mathcal{SD}$ since the set $\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\}$ contains, in the worse case $|\varphi^*(d)|$.

We do not talk of connecting paths, since the walk $CW(d)$ is not always a path (see [16] and Figure 2). If $CW(d)$ is a path it connects the vertex $\sigma^*(d)$ to the vertex $\sigma^*(\varphi^n(d))$ where d and $\varphi^n(d)$ belong to $\mathcal{SD}$.

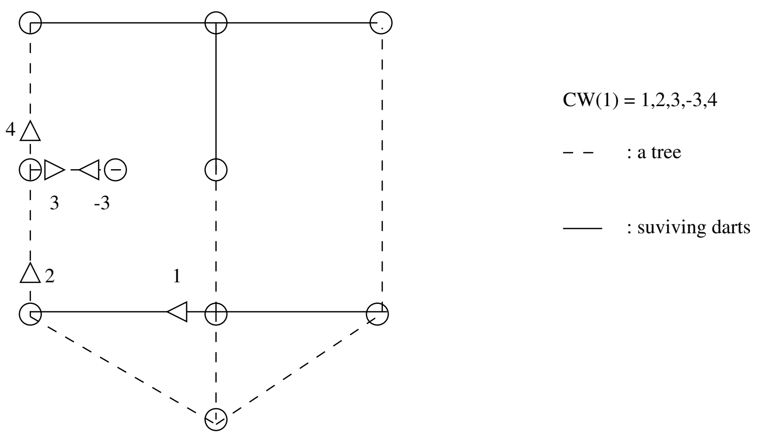

Figure 2: *A connecting walk which is not a path*

Proposition 4 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a contraction kernel K, the set of non-surviving darts of a connecting walk is included in exactly one connected component of K:*

$$\forall d \in \mathcal{SD} \quad CW(d) - \{d\} = \emptyset \textit{ or } \exists! \mathcal{T} \in \mathcal{CC}(\mathcal{T}) \quad | \quad CW(d) - \{d\} \subset \mathcal{T}$$

Proof:

Let us consider $d \in \mathcal{SD}$ and:

$$CW(d) = d, \varphi(d), \ldots, \varphi^{n-1}(d) \text{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\}$$

If $n = 1$ we have $CW(d) = d$ and $CW(d) - \{d\} = \emptyset$. Otherwise, $\varphi(d)$ is not a surviving dart, and there must exist one $\mathcal{T}$ such that $\varphi(d) \in \mathcal{T}$. Let us consider the last dart in the sequence $\varphi^j(d)$ of darts in $\mathcal{T}$:

$$p = Max\{k \in \{1, \ldots, n-1\} \quad | \quad \forall j \in \{1, \ldots, k\} \quad \varphi^j(d) \in \mathcal{T}\}$$

We have at least $p = 1$, let us suppose that $p < n - 1$. Since $\varphi^{p+1}(d)$ is not a surviving dart, there must exist another set $\mathcal{T}'$ such that $\varphi^{p+1}(d) \in \mathcal{T}'$. But, $\mathcal{T}$ being symmetric, we have $\alpha(\varphi^p(d)) \in \mathcal{T}$. Since $\varphi^{p+1}(d) = \sigma(\alpha(\varphi^p(d)))$ we have:

$$\varphi^{p+1}(d) \in \sigma^*(\mathcal{T}) \cap \sigma^*(\mathcal{T}')$$

Which is forbidden by the definition of a disjoint vertex-set (see definition 2). $\square$

Definition 11 Set of Connecting Walks

Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a contraction kernel K with surviving darts $\mathcal{SD}$, the set of all connecting walks will be denoted by:

$$\boldsymbol{\mathcal{D}}_K = \{CW(d) \quad | \quad d \in \mathcal{SD}\}$$

Proposition 5 *Given a connected combinatorial map G and a contraction kernel K, the application:*

$$CW \left(\begin{array}{lcl} \mathcal{SD} & \to & \boldsymbol{\mathcal{D}}_K \\ d & \mapsto & CW(d) \end{array} \right.$$

is bijective.

Proof:

This application is trivially surjective since the set of connecting walks is generated from the set of surviving darts. Moreover, each connecting walk containing only one surviving dart the application is trivially injective and thus bijective. $\square$

Proposition 6 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and a contraction kernel K each dart of $\boldsymbol{\mathcal{D}}$ belongs to exactly one connecting walk:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \exists! d' \in \mathcal{SD} \mid d \in CW(d')$$

Proof:

By definition, each connecting walk contains only one dart in $\mathcal{SD}$, thus if d belongs to $\mathcal{SD}$, $CW(d)$ exists and is unique.

Now let us consider $d \in K$. According to proposition 3 we have: $\varphi^*(d) \cap \mathcal{SD} \neq \emptyset$. Let us consider:

$$d' \in \varphi^*(d) \cap \mathcal{SD} \mid d' = \varphi^{-n}(d) \text{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^{-p}(d) \in \mathcal{SD}\}$$

we have obviously $d \in CW(d')$. Let us suppose that we can find another dart $d'' \in \mathcal{SD}$ such that $d \in CW(d'')$. Then $d'' = \varphi^{-p}(d)$ with $p > n$. Thus the walk:

$$\begin{array}{rcllll} CW(d'') & = & d'', & \varphi(d''), \dots, & \varphi^{p-n}(d''), \dots & , \varphi^{p}(d''), \dots \\ & = & d'' & , \dots, & d', \dots & , d, \dots \end{array}$$

contains at least the two darts d'' and d' in $\mathcal{SD}$, which is forbidden by the definition of a connecting walk. $\square$

Definition 12 Reversal of Connecting walks

Given a connected combinatorial map and a contraction kernel K, the opposite permutation α_K from $\boldsymbol{\mathcal{D}}_K$ to itself maps each connecting walk $CW(d)$ with $d \in \mathcal{SD}$ to $CW(\alpha(d))$:

$$\alpha_K \left(\begin{array}{lcl} \boldsymbol{\mathcal{D}}_K & \rightarrow & \boldsymbol{\mathcal{D}}_K \\ CW(d) & \mapsto & CW(\alpha(d)) \end{array} \right.$$

Remark 1 *The function which associates to each dart its connecting walk and the permutation α being bijective α_K is bijective. It is thus a permutation on $\boldsymbol{\mathcal{D}}_K$. Moreover,*

$$\alpha_K \circ \alpha_K(CW(d)) = CW(\alpha \circ \alpha(d)) = CW(d)$$

α_K is an involution.

Lemma 2 *Given a connected combinatorial map and a contraction kernel K, the application*

$$follow\left(\begin{array}{lll} \mathcal{SD} & \to & \mathcal{SD} \\ d & \mapsto & \varphi^n(d) \text{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}\} \end{array}\right.$$

is bijective.

Proof:

Note that according to the previous notations we have:

$$CW(d) = d, \dots, \varphi^{n-1}(d)$$

The connecting walk $CW(d)$, and thus $\varphi^n(d)$, is defined for all darts in $\mathcal{SD}$. Now let us suppose that we can find two darts d and d' such that $follow(d) = follow(d')$. Then there exist two integers n, p, with $n \geq p$ such that $\varphi^n(d) = \varphi^p(d')$. Thus we have $d' = \varphi^{n-p}(d) \in \mathcal{SD}$. The integer n being the minimal integer different from zero which realizes this equality we have $n = p$ and thus $d = d'$. $\square$

Proposition 7 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a contraction kernel K the application $follow \circ \alpha$ maps $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$ into $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$ for all $\mathcal{T} \in \mathcal{CC}(K)$:*

$$\forall \mathcal{T} \in \mathcal{CC}(K), \quad \forall d \in \sigma^*(\mathcal{T}) \cap \mathcal{SD} \quad follow(\alpha(d)) \in \sigma^*(\mathcal{T}) \cap \mathcal{SD}$$

Proof:

Given a dart d in $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$, we have :

$$CW(\alpha(d)) = \alpha(d), \sigma(d), \dots, \varphi^{n-1}(\alpha(d))$$

If $n = 1$, we have $follow(\alpha(d)) = \sigma(d)$ which belongs to $\sigma^*(\mathcal{T})$ by hypothesis. Otherwise, by definition of a connecting walk $\sigma(d)$ does not belong to $\mathcal{SD}$. Since, we have by definition of a contraction kernel:

$$\sigma^*(d) \subset \mathcal{T} \cup \mathcal{SD}$$

we must have $\sigma(d) \in \mathcal{T}$. Thus we have thanks to proposition 4:

$$\{\sigma(d), \dots, \varphi^{n-1}(\alpha(d))\} \subset \mathcal{T}$$

The tree $\mathcal{T}$ being symmetric we have: $\alpha(\varphi^{n-1}(\alpha(d))) \in \mathcal{T}$. Thus :

$$follow(\alpha(d)) = \varphi^n(\alpha(d)) = \sigma(\alpha(\varphi^{n-1}(\alpha(d))) \in \sigma^*(\mathcal{T})$$

□

The application $follow$ being bijective, it can be considered as a permutation on the set of surviving darts. Moreover, the vertices $\sigma^*(d) \subset \sigma^*(\mathcal{T}) \cap \mathcal{SD}$ may be interpreted as the leafs of the tree $\mathcal{T}$. Thus the orbits of the permutation $follow \circ \alpha$ describe the leafs of the trees.

Proposition 8 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a contraction kernel K, the applications:*

$$\varphi_K \left(\begin{array}{lll} \boldsymbol{\mathcal{D}}_K & \to & \boldsymbol{\mathcal{D}}_K \\ CW(d) = d, \varphi(d), \dots, \varphi^{n-1}(d) & \mapsto & CW(\varphi^n(d)) \end{array} \right.$$

and $\sigma_K = \varphi_K \circ \alpha_K$ define two permutations on $\boldsymbol{\mathcal{D}}_K$.

Proof:

We have $\varphi_K(CW(d)) = CW(follow(d))$ thus:

$$\varphi_K = CW \circ follow \circ CW^{-1}$$

The application φ_K is bijective as the composition of bijective applications.

Moreover, the application σ_K is the composition of two permutations and is thus a permutation. □

Proposition 9 *Given a connected combinatorial map $G = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ and a contraction kernel K, the connecting walks of two consecutive surviving darts in a given σ-orbit are consecutive in a σ_K-orbit:*

$$\forall d \in \mathcal{SD} \quad \sigma(d) \in \mathcal{SD} \Rightarrow \sigma_K(CW(d)) = CW(\sigma(d))$$

Proof:

We have $\varphi(\alpha(d)) = \sigma(d) \in \mathcal{SD}$. Thus the connecting walk $CW(\alpha(d))$ is reduced to d and we have $follow(\alpha(d)) = \sigma(d)$. Thus:

$$\sigma_K(CW(d)) = CW(follow(\alpha(d))) = CW(\sigma(d))$$

□

Corollary 1 *Given a connected combinatorial map and a contraction kernel K, if a σ-orbit is included in $\mathcal{SD}$, the connecting walks of any two consecutive darts within this σ-orbit, are consecutive in a σ_K-orbit:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \sigma^*(d) \subset \mathcal{SD} \Rightarrow \forall d' \in \sigma^*(d) \quad \sigma_K(CW(d')) = CW(\sigma(d'))$$

Corollary 2 *With the same hypothesis as proposition 9, if a σ-orbit is included in $\mathcal{SD}$, the application CW maps this σ-orbit to a σ_K-orbit. Moreover two consecutive darts in the σ-orbit are mapped into two consecutive connecting walks in the σ_K-orbit:*

$$\forall d \in \boldsymbol{\mathcal{D}} \quad \sigma^*(d) \subset \mathcal{SD} \Rightarrow \sigma_K^*(CW(d)) = CW(\sigma^*(d))$$

Proof:

A basic recursion on the power of $\sigma_K^i(CW(d))$ $\square$

The proposition 9 and corollaries 1 and 2 show that the permutation σ_K may be immediately deduced from the permutation σ for surviving darts. Intuitively, this last property means that one vertex which does not belong to any tree, will not be affected by contractions. Let us now study the σ_K-orbit of contracted vertices:

Proposition 10 *Given a connected combinatorial map, a contraction kernel K and a tree $\mathcal{T}$ in $\mathcal{CC}(K)$. The σ_K-orbit of any connecting walk defined by a dart d in $\sigma^*(T) \cap \mathcal{SD}$ is equal to $CW(\sigma^*(\mathcal{T}) \cap \mathcal{SD})$:*

$$\forall \mathcal{T} \in \mathcal{CC}(K), \quad \forall d \in \sigma^*(\mathcal{T}) \cap \mathcal{SD} \quad CW^{-1}(\sigma_K^*(CW(d))) = \sigma^*(\mathcal{T}) \cap \mathcal{SD}$$

Proof:

Let us first show that:

$$\forall \mathcal{T} \in \mathcal{CC}(K), \quad \forall d \in \sigma^*(\mathcal{T}) \cap \mathcal{SD} \quad CW^{-1}(\sigma_K^*(CW(d))) \subset \sigma^*(\mathcal{T}) \cap \mathcal{SD}$$

Let us write the σ_K-orbits of $CW(d)$ as:

$$(CW(d_0) = CW(d), CW(d_1), \ldots, CW(d_n)).$$

We have to show that:

$$\forall i \in \{0, \ldots, n\} \quad CW^{-1}(CW(d_i)) = d_i \in \sigma^*(\mathcal{T}) \cap \mathcal{SD}$$

The proposition is true for $i = 0$. Let us suppose that the proposition is true for all $k \in \{0, \ldots, i\}$. We have:

$$\begin{aligned} CW^{-1}(CW(d_{i+1}))) &= CW^{-1}(\sigma_K(CW(d_i))) \\ &= CW^{-1}(\varphi_K(\alpha_K(CW(d_i)))) \\ &= CW^{-1}(\varphi_K(CW(\alpha(d_i)))) \\ &= CW^{-1}(CW(follow(\alpha(d_i)))) \\ &= follow(\alpha(d_i)) \end{aligned}$$

We know, thanks to proposition 7 that $follow(\alpha(d_i))$ belongs to $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$. Thus d_{i+1} belongs to the same set, and the recursive hypothesis holds until $i + 1$.

Thus:

$$CW^{-1}(\sigma_K^*(CW(d))) \subset \sigma^*(\mathcal{T}) \cap \mathcal{SD}$$

Conversely, let us consider the submap $G' = (\mathcal{T}, \sigma', \alpha)$ of G. Since $\mathcal{T}$ is a tree, we have: $\mathcal{T} = \varphi'^*(d_1) = (d_1, \ldots, d_m)$ for a given $d_1 \in \mathcal{T}$. Using proposition 6 we know that each dart belongs to only one connecting walk. Thus,

$$\forall i \in \{1, \ldots, m\} \quad \exists! d'_i \in \mathcal{SD} \mid d_i \in CW(d'_i)$$

Let us show that:

$$\forall k \in \{1, \ldots, m\} \quad CW(\alpha(d'_{k+1})) \in \sigma_K^*(CW(\alpha(d'_k))) \tag{1}$$

If $d_{k+1} = \varphi'(d_k) = \varphi(d_k) \in \mathcal{T}$, we have $d_{k+1} \in CW(d'_k)$ and thus $d'_{k+1} = d'_k$.

Otherwise, we have:

$$d_{k+1} = \varphi'(d_k) = \sigma^p(\alpha(d_k)) \text{ with } \forall k \in \{1, \ldots, p-1\} \quad \sigma^k(\alpha(d_k)) \notin \mathcal{T}$$

Since $\sigma^*(\alpha(d_k)) \subset \mathcal{T} \cup \mathcal{SD}$, we have:

$$\forall k \in \{1, \ldots, p-1\} \quad \sigma^k(\alpha(d_k)) \in \mathcal{SD}$$

Moreover, since $\varphi'(d_k) \neq \varphi(d_k)$, we have $\varphi(d_k) = \sigma(\alpha(d_k)) \notin \mathcal{T}$ and thus $\sigma(\alpha(d_k)) \in \mathcal{SD}$. This last property is equivalent to $p > 1$. Moreover, we have by definition of the function $follow$, $follow(d'_k) = \sigma(\alpha(d_k))$, thus:

$$\begin{aligned} \varphi_K(CW(d'_k)) &= CW(follow(d'_k)) \\ \Rightarrow \sigma_K(CW(\alpha(d'_k))) &= CW(\sigma(\alpha(d_k))) \end{aligned} \tag{2}$$

In the same way, we have: $\sigma^{p-1}(\alpha(d_k)) \in \mathcal{SD}$ and $\varphi(\alpha(\sigma^{p-1}(-d_k))) = \sigma^p(\alpha(d_k))$

Thus:

$$d_{k+1} = \sigma^p(\alpha(d_k)) \in CW(\alpha(\sigma^{p-1}(\alpha(d_k))))$$

Therefore, using proposition 6: $d'_{k+1} = \alpha(\sigma^{p-1}(\alpha(d_k)))$. Using proposition 9 we have:

$$CW(\alpha(d'_{k+1})) = CW(\sigma^{p-1}(\alpha(d_k))) \in \sigma_K^*(CW(\sigma(\alpha(d_k))))$$

Therefore, using equation 2:

$$CW(\alpha(d'_{k+1})) \in \sigma_K^*(\sigma_K(CW(\alpha(d'_k)))) = \sigma_K^*(CW(\alpha(d'_k)))$$

We have thus:

$$\forall k \in \{1, \ldots, m\} \quad CW(\alpha(d'_{k+1})) \in \sigma_K^*(CW(\alpha(d'_k)))$$

Since d belongs to $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$, its σ-orbit intersects $\mathcal{T}$:

$$\exists p \in \mathbb{N}^* \mid \sigma^p(d) \in \mathcal{T} \text{ and } \forall k \in \{1, \ldots, p-1\} \quad \sigma^k(d) \in \mathcal{SD}$$

If $p = 1$, we have $CW(\alpha(d)) = \alpha(d), b_1, \ldots, b_r$ with $b_1 = \varphi(\alpha(d)) = \sigma(d) \in \mathcal{T}$. Thus it exists one k in $\{1, \ldots, m\}$ such that $d = \alpha(d'_k)$. We have thus:

$$CW(d) \in \sigma_K^*(CW(\alpha(d'_k))) \Longleftrightarrow \sigma_K^*(CW(d)) = \sigma_K^*(CW(\alpha(d'_k)))$$

Otherwise, we have: $CW(\alpha(\sigma^{p-1}(d))) = \alpha(\sigma^{p-1}(d)), \sigma^p(d), \ldots$ with $\sigma^p(d) \in \mathcal{T}$. Therefore, using proposition 6, we have:

$$\exists d'_k \in \mathcal{SD} \text{ with } k \in \{1, \ldots, m\} \mid d'_k = \alpha(\sigma^{p-1}(d))$$

Moreover, using proposition 9 we have:

$$\sigma_K^{p-1}(CW(d)) = CW(\sigma^{p-1}(d))$$

Thus

$$CW(d) \in \sigma_K^*(CW(\sigma^{p-1}(d))) = \sigma_K^*(CW(\alpha(d'_k)))$$

In the same way, given a dart d' in $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$, we have:

$$\exists j \in \{1, \ldots, m\} \mid CW(d') \in \sigma_K^*(CW(\alpha(d'_j)))$$

Using equation 1, we have:

$$CW(d') \in \sigma_K^*(CW(\alpha(d'_j))) = \sigma_K^*(CW(\alpha(d'_k))) = \sigma_K^*(CW(d))$$

Therefore:

$$\forall d' \in \sigma^*(\mathcal{T}) \cap \mathcal{SD} \quad d' \in CW^{-1}(\sigma_K^*(CW(d)))$$

Which is equivalent to:

$$\sigma^*(\mathcal{T}) \cap \mathcal{SD} \subset CW^{-1}(\sigma_K^*(CW(d)))$$

The equality between the two set is thus demonstrated. $\square$

Proposition 10 shows that if a connecting walk traverse a given tree its σ_K-orbit is equal to the set of connecting walks traversing the same tree. Therefore, given any tree $\mathcal{T}$ in $\mathcal{CC}(K)$, all the connecting walks in $CW(\sigma^*(T) \cap \mathcal{SD})$ belong to a same σ_K-orbit and are thus ordered. Moreover, the application CW being bijective, the order defined on $CW(\sigma^*(T) \cap \mathcal{SD})$ induce an order on the set of surviving darts adjacent to $\mathcal{T}$: $\sigma^*(\mathcal{T}) \cap \mathcal{SD}$.

Corollary 3 *Given a connected combinatorial map G, a contraction kernel K, and a non-surviving dart d in K. The opposite of the two connecting walks $CW(d')$ and $CW(d'')$ including d and $\alpha(d)$:*

$$\left\{ \begin{array}{lll} d & \in & CW(d') \\ \alpha(d) & \in & CW(d'') \end{array} \right.$$

belong to the same σ_K-orbit:

$$CW(\alpha(d'')) \in \sigma_K^*(CW(\alpha(d')))$$

Proof:

The existence and the uniqueness of darts d' and d'' is provided by proposition 6. By definition of a contraction kernel it exists a unique tree $\mathcal{T}$ in $\mathcal{CC}(K)$ such that $\alpha^*(d) \subset \mathcal{T}$. Using proposition 4 we have:

$$\begin{array}{lll} CW(d') - \{d'\} & \subset & \mathcal{T} \\ CW(d'') - \{d''\} & \subset & \mathcal{T} \end{array}$$

Thus:

$$\begin{array}{lll} \sigma(\alpha(d')) = \varphi(d') \in CW(d') - \{d'\} & \subset & \mathcal{T} \\ \sigma(\alpha(d'')) = \varphi(d'') \in CW(d'') - \{d''\} & \subset & \mathcal{T} \end{array}$$

Therefore, we have $\alpha(\{d', d''\}) \subset \sigma^*(\mathcal{T}) \cap \mathcal{SD}$. Using proposition 10 we have:

$$\alpha(d'') \in CW^{-1}(\sigma_K^*(CW(\alpha(d')))) \Longleftrightarrow CW(\alpha(d'')) \in \sigma_K^*(CW(\alpha(d')))$$

$\square$

This last corollary shows that the opposite of two connecting walks containing two opposite darts belong to the same σ_K-orbit.

2.5 Link between connecting walks and contraction

This short section show that the connecting walk map (see definition 13 bellow) is isomorph to the contracted map defined by the contraction kernel. Thus, all the properties defined in the connecting walk map may be extended to the contracted one.

Definition 13 Connecting walk map *Given a connected combinatorial map G and a contraction kernel K, the* **connecting walk map** *associated to G and K is denoted GC and is defined by:*

$$GC = (\boldsymbol{\mathcal{D}}_K, \sigma_K = \varphi_K \circ \alpha_K, \alpha_K)$$

Theorem 3 *Given a connected combinatorial map G and a contraction kernel K, the connecting walk map GC is isomorph to the contracted map $G' = G/K$:*

$$GC \cong G/K$$

Proof:

We have:

$$\begin{array}{rcl} GC & = & (\boldsymbol{\mathcal{D}}_K, \sigma_K, \alpha_K) \\ G' & = & (\boldsymbol{\mathcal{D}} - K, \sigma', \alpha) = (\mathcal{SD}, \sigma', \alpha) \end{array}$$

Now, let us consider the application $\phi = (\chi, CW)$ from G' to GC such that:

$$\chi : \begin{array}{rcl} \sigma' & \mapsto & \sigma_K \\ \alpha & \mapsto & \alpha_K \end{array}$$

Since the application CW is bijective (see proposition 5), ϕ is bijective. Let us show that it is a morphism, thus that:

$$\forall d \in \mathcal{SD} \left\{ \begin{array}{rcl} CW(\alpha(d)) & = & \alpha_K(CW(d)) \\ CW(\sigma'(d)) & = & \sigma_K(CW(d)) \end{array} \right.$$

The first equality is given by the definition of the involution α_K. Moreover we have:

$$\begin{aligned}\sigma_K(CW(d)) &= \varphi_K \circ \alpha_K(CW(d)) \\ &= \varphi_K(CW(\alpha(d))) \\ &= CW(follow(\alpha(d)))\end{aligned}$$

The application CW being bijective, the second equality will be demonstrated iff we show that $\sigma'(d) = follow(\alpha(d))$.

We have, by definition $G' = \overline{\overline{G} \setminus K}$. Thus $\sigma' = \varphi' \circ \alpha$ with $\varphi' = \varphi \circ p_{\mathcal{D},\mathcal{SD}}$. Thus $\sigma'(d) = \varphi \circ p_{\mathcal{D},\mathcal{SD}}(\alpha(d)) = \varphi^n(\alpha(d))$ with:

$$n = Min\{p \in \mathbb{N}^* \mid \varphi^p(\alpha(d)) \in \mathcal{SD}\} \tag{3}$$

Moreover, according to Lemma 2, we have $follow(\alpha(d)) = \varphi^n(\alpha(d))$ with n satisfying equation 3. Thus:

$$\sigma'(d) = follow(\alpha(d)) \Rightarrow CW(\sigma'(d)) = \sigma_K(CW(d))$$

□

3 Equivalent Contractions Kernels

This section is devoted to the application of successive parallel contractions. Each set of contractions is defined by a contraction kernel. We show in this section that applying successively two contraction kernels is equivalent to applying a bigger one only once (see section 3.1). Conversely, we show that a contraction kernel may be decomposed into two smaller ones. The successive application of the resulting contraction kernels is equivalent to the application of the initial one (see section 3.2)

Definition 14 Inclusion of Contraction Kernels

Let us consider a combinatorial map G_1, and two contraction kernels K_1 and K_2 defined on G_0. We will say that the contraction kernel K_2 includes K_1 iff $K_1 \subset K_2$. In this case each connected component of K_1 is included in exactly one connected component of K_2:

$$\forall \mathcal{T} \in \mathcal{CC}(K_1) \quad \exists!\ \mathcal{T}' \in \mathcal{CC}(K_2) \mid \mathcal{T} \subset \mathcal{T}'$$

Definition 15 Predecessor and Successor Kernels

Given a combinatorial map $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$*, a contraction kernel* K_1 *of* G_0 *and the contracted combinatorial map* $G_1 = G_0/K_1$*. If* K_2 *is a contraction kernel of* G_1 *then we say that* K_1 *is the predecessor of* K_2*, or that* K_2 *is the successor of* K_1*. This relation will be denoted* $K_1 \prec K_2$*.*

The successive application of K_1 *and* K_2 *forms a new operator on* G_0 *denoted by* $K_2 \circ K_1$*.*

Lemma 3 *Given a combinatorial map* $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ *and two disjoint forests of* $\boldsymbol{\mathcal{D}}$*,* F_1 *and* F_2*. If* $F_1 \cup F_2$ *is a forest, we can contract* G_0 *in different ways, but the final combinatorial map is always the same:*

$$G_0/(F_1 \cup F_2) = (G_0/F_1)/F_2$$

Proof:

This property may be trivially deduced from the commutativity of contraction operations. □

This lemma shows that the final combinatorial map does not depend on the order of the contractions. Thus, given two successive contraction kernels K_1 and K_2, if a contraction kernel K' may be deduced from $K_1 \cup K_2$, the application of K' will be equivalent to the successive application of K_1 and K_2. In the same way, given two contraction kernels K_1 and K' such that $K_1 \subset K'$ if we can define a contraction kernel K_2 on the set of darts $K' - K_1$, the successive application of K_1 and K_2 is equivalent to the application of K'.

3.1 Deriving an Inclusion Kernel from Successor Kernels

This section is devoted to the demonstration of Theorem 4 which shows that, given two successive contraction kernels K_1 and K_2 of a combinatorial map G_0, with $K_1 \prec K_2$, we can define a third contraction kernel K_2' such that: $(G_0/K_1)/K_2 = G_0/K_2'$. The following notations will be used in this section (see Figure 3):

- $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$ denotes the initial combinatorial map.
- K_1 and K_2 denote the two successive contraction kernels such that $K_1 \prec K_2$. Note that we have $K_1 \subset \boldsymbol{\mathcal{D}}$ and $K_2 \subset \boldsymbol{\mathcal{D}}$.

- $G_1 = (\mathcal{SD}_1, \sigma_1, \alpha)$ denotes the contracted combinatorial map G_0/K_1.
- $G' = (K_2 \cup K_1, \sigma', \alpha)$ and $G'_1 = (K_2, \sigma'_1, \alpha)$ denote respectively, two submaps of G_0 and G_1. The combinatorial maps G'_1 and G' are based on respectively the darts that are contracted by K_2 on G_1 and the darts that are contracted by the successive applications of K_1 and K_2 on G_0.

$$\begin{tikzcd}
G'' = (K_1, \sigma'', \alpha) \arrow[r, phantom, "\subset"] & G' = (K_2 \cup K_1, \sigma', \alpha) \arrow[r, phantom, "\subset"] & G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha) \arrow[d, "K_1"] \\
 & G'_1 = (K_2, \sigma'_1, \alpha) \arrow[r, phantom, "\subset"] & G_1 = (\mathcal{SD}_1, \sigma_1, \alpha) \arrow[d, "K_2"] \\
 & & G_2 = (\mathcal{SD}_2, \sigma_2, \alpha)
\end{tikzcd}$$

Figure 3: *Combinatorial maps that will be used in this section. The contractions are represented by arrows.*

We know, by definition of the contraction kernel K_2 that G'_1 is a forest of G_1. One of the aims of this section is to show that G' is also a forest of G_0. Note that, G'_1 is not a submap of G' besides the fact that $K_2 \subset K_2 \cup K_1$. Indeed, G'_1 is a submap of G_1, since G_1 is not a sub-map of G', G'_1 is not a sub-map of G'. The relation between the combinatorial map G'_1 and G' is given by equation 5.

Lemma 4 *Using previously defined notations, the following relations hold:*

1. $K_2 \subset \mathcal{SD}_1$ *and* $K_2 \neq \mathcal{SD}_1$
2. $K_1 \cap K_2 = \emptyset$
3. $(K_2 \cup K_1) \cap \mathcal{SD}_1 = K_2$

4. $\boldsymbol{\mathcal{D}} - K_2 \cup K_1 = \mathcal{SD}_1 - K_2$

Proof:

1. The contraction kernel K_2 being defined on $G_1 = (\mathcal{SD}_1, \sigma_1, \alpha)$, we have, by definition of a contraction kernel:

$$K_2 \subset \mathcal{SD}_1 \text{ and } K_2 \neq \mathcal{SD}_1$$

2. We have:

$$K_2 \subset \mathcal{SD}_1 = \boldsymbol{\mathcal{D}} - K_1$$

Thus:

$$K_2 \cap K_1 = \emptyset$$

3. This is a consequence of the two preceding equalities:

$$\begin{array}{rcl} K_2 \cup K_1 \cap \mathcal{SD}_1 & = & (K_1 \cup K_2) \cap \mathcal{SD}_1 \\ & = & (K_1 \cap \mathcal{SD}_1) \cup (K_2 \cap \mathcal{SD}_1) \\ & = & K_2 \end{array}$$

4. We have:

$$\begin{array}{rcl} \boldsymbol{\mathcal{D}} - K_2 \cup K_1 & = & \boldsymbol{\mathcal{D}} - (K_1 \cup K_2) \\ & = & (\boldsymbol{\mathcal{D}} - K_1) - K_2 \\ & = & \mathcal{SD}_1 - K_2 \end{array}$$

□

Note that lemma 4 is demonstrated for two succesive contraction kernel. Nevertheless, this lemma using only quite general properties of contraction kernels it remains true in the following cases:

- K_1 and K_2 are both dual contraction kernels.
- K_1 is a contraction kernel and K_2 is a dual one.
- K_2 is a contraction kernel and K_1 is a dual one.

The following lemmata (5, 6, 7 and 8) establish a link between the φ-orbits of the combinatorial map G' and the ones of the combinatorial map G_1'. The φ-orbits of G_1' being deduced from the ones of G_1 which is isomorphic to the connecting walk map of G_0, we will first, study the connections between the φ-orbits of G' and the connecting walks of G_0.

Lemma 5 *Given a combinatorial map $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and two contraction kernels K_1 and K_2 such that $K_1 \prec K_2$. The sub map $G' = (K_2 \cup K_1, \sigma', \alpha)$ verifies:*

1. *Each connecting walk of K_1 defined by a dart d in K_2 is included in the φ'-orbit: $\varphi'^*(d)$. Moreover the same order on the elements applies in $CW(d)$ and $\varphi'^*(d)$:*

$$\forall d \in K_2 \quad CW(d) \subset \varphi'^*(d)$$

2. *Each non-surviving dart of K_1 belongs to a connecting walk. All the non surviving darts of this connecting walk appear in a φ'-orbit of G' in the same order as in the connecting walk:*

$$\forall d \in K_1 \quad \exists! d' \in \mathcal{SD}_1 \mid d \in CW(d') - \{d'\} \subset \varphi'^*(d)$$

Where, the connecting walks are defined in G_0 by K_1 and φ' denotes the permutation φ in the sub map G'.

Proof:

1. Let us consider d in K_2. Since $K_2 \subset \mathcal{SD}_1$, d is a surviving dart and we can consider the connecting walk:

   $$CW(d) = d, \varphi(d) \ldots, \varphi^{n-1}(d) \text{ with } n = Min\{p \in \mathbb{N}^* \mid \varphi^p(d) \in \mathcal{SD}_1\}$$

   If $n = 1$, $CW(d)$ is reduced to d and is thus included in $\varphi'^*(d)$.

   Otherwise, let us consider the following proposition:

   $$\forall j \in \{0, \ldots, i\} \text{ with } i < p \quad \varphi'^j(d) = \varphi^j(d)$$

   This proposition is true at least for $i = 0$ since $\varphi'^0(d) = \varphi^0(d) = d$. Let us suppose that the proposition is true for a given $i < p$.

   We have by definition of the restriction operator:

   $$\begin{array}{lcl} \varphi'^{i+1}(d) & = & \sigma^q(\alpha(\varphi'^i(d))) \text{ with} \\ q & = & Min\{k \in \mathbb{N}^* \mid \sigma^k(\alpha(\varphi'^i(d))) \in K_2 \cup K_1\} \end{array}$$

   Since $\sigma(\alpha(\varphi'^i(d))) = \sigma((\alpha(\varphi^i(d))) = \varphi^{i+1}(d) \in K_1 \subset K_2 \cup K_1$ we have $q = 1$ and the recurrence hypothesis hold until $i + 1$.

2. If $d \in K_1$ we know, thanks to proposition 6 that:

$$\exists! d' \in \mathcal{SD}_1 \mid d \in CW(d')$$

Moreover, since $d \neq d'$, $CW(d')$ is not reduced to d'. Let $CW(d') = d', \varphi(d'), \dots, \varphi^{p-1}(d')$. Since $CW(d') - \{d'\}$ is not empty we must have $p > 1$. Moreover, by definition of a connecting walk:

$$\forall l \in \{1, \dots, p-1\} \quad \varphi^l(d) \in K_1 \subset K_2 \cup K_1$$

Thus, as previously: $\forall l \in \{1, \dots, p-1\}$ $\varphi^l(d) = \varphi'^l(d)$. Therefore,

$$CW(d') - \{d'\} \subset \varphi'^*(\varphi(d')) = \varphi'^*(d)$$

□

Each connecting walk is by definition, included in one φ-orbit. The last lemma only shows that this property remains true in the combinatorial map G' for the connecting walks defined in G_0 by K_1. Intuitively, this proposition is due to the fact that restricting a combinatorial map enlarge the set of darts of each face. Thus a connecting walk included in one φ-orbit $\varphi^*(d)$ with d in $K_2 \cup K_1$ will be included in $\varphi'^*(d)$.

Lemma 6 *Given a combinatorial map $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and two contraction kernels K_1 and K_2 such that $K_1 \prec K_2$. For all d in K_2, the φ'-orbit of d: $\varphi'^*(d)$, in $G' = (K_2 \cup K_1, \sigma', \alpha)$ may be expressed by (see Figure 4):*

$$\varphi'^*(d) = d_1, \dots, d_2, \dots, d_m \dots \text{ with } \{d_1, \dots, d_m\} \subset K_2$$

Moreover, for each i in $\{1, \dots, m\}$ it exists a serie of darts $(d_i^1, \dots, d_i^{n_i})$ in $\mathcal{SD}_1 - K_2$ such that:

$$\begin{array}{lcl} \varphi'^*(d) & = & P_1.P_2. \dots, P_m \text{ with} \\ P_i & = & CW(d_i).(CW(d_i^1) - \{d_i^1\}) \dots (CW(d_i^{n_i}) - \{d_i^{n_i}\}) \end{array}$$

In other words, any face of G' which contains at least one dart in K_2 may be considered as a concatenation of connecting walks, without the darts belonging to $\mathcal{SD}_1 - K_2$.

Proof:

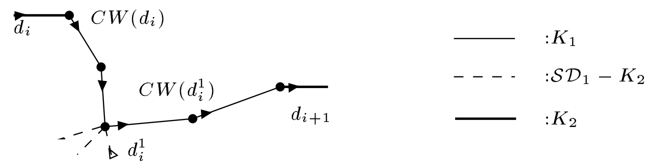

Figure 4: *One P_i defined by two connecting walks*

Since d belongs to K_2 we know that the intersection between $\varphi'^*(d)$ and K_2 is not empty. We can thus suppose the existence of the darts $\{d_1, \ldots, d_m\}$ with m at least 1. Moreover, the demonstration being the same for each P_i it is sufficient to show it for a given i and to show that the permutation φ' maps the last dart of P_i to d_{i+1}.

Using Lemma 5 we have: $CW(d_i) \subset \varphi'^*(d_i) = \varphi'^*(d)$. Let us denote $CW(d_i) = b_1, \ldots, b_r$. We have $\varphi(b_r) = follow(d_i) = \varphi_1(d_i)$. Moreover,

$$\begin{array}{l} \varphi'(b_r) = \sigma^p((\alpha(b_r)) \text{ with} \\ \forall k \in \{1, \ldots, p-1\} \quad \sigma^k(\alpha(b_r)) \in \boldsymbol{\mathcal{D}} - K_2 \cup K_1 = \mathcal{SD}_1 - K_2 \end{array}$$

- If $\varphi'(b_r) \in K_2$ we have according to our notations $\varphi'(b_r) = d_{i+1}$ and $P_i = CW(d_i)$.
- If $p = 1$, we have $\varphi'(b_r) = \varphi(b_r) = \varphi_1(d_i)$. Moreover, $\varphi'(b_r) \in K_2 \cup K_1$ and $\varphi_1(d_i) \in \mathcal{SD}_1$, thus $\varphi'(b_r) \in (K_2 \cup K_1) \cap \mathcal{SD}_1 = K_2$ (see Lemma 4). Thus as previously, $\varphi'(b_r) = d_{i+1}$ and $P_i = CW(d_i)$.

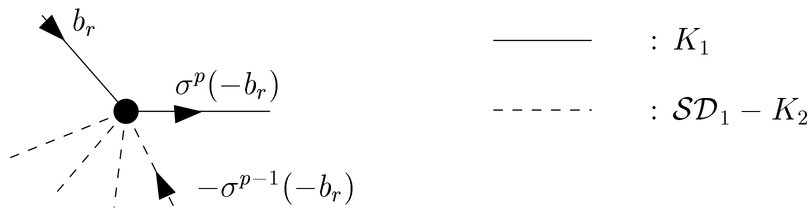

Figure 5: *A zoom on a connection between two connecting walks*

- Otherwise, we have $p > 1$ and $\varphi'(b_r) \in K_1$ (see Figure 5). In this case we have: $\sigma^{p-1}(\alpha(b_r)) \in \mathcal{SD}_1 - K_2$ and $\varphi(\alpha(\sigma^{p-1}(\alpha(b_r)))) = \sigma^p(\alpha(b_r))$, Thus:
$$\varphi'(b_r) = \sigma^p(\alpha(b_r)) \in CW(\alpha(\sigma^{p-1}(\alpha(b_r))))$$

Using Lemma 5 we have:

$$CW(\alpha(\sigma^{p-1}(\alpha(b_r)))) - \{\alpha(\sigma^{p-1}(\alpha(b_r)))\} \subset \varphi'^*(b_r) = \varphi'^*(d)$$

If we denote $d_i^1 = \alpha(\sigma^{p-1}(\alpha(b_r)))$ we obtain: $P_i = CW(d_i).(CW(d_i^1) - \{d_i^1\}) \ldots$.

Let us suppose, that P_i can be written as:

$$P_i = CW(d_i).(CW(d_i^1) - \{d_i^1\}) \ldots (CW(d_i^j) - \{d_i^j\}) \ldots$$

for a given j. Let us denote $CW(d_i^j)$ by:

$$CW(d_i^j) = b'_1, \ldots, b'_{r'}$$

we have $\varphi'(b'_{r'}) = \sigma^{p'}(\alpha(b_{r'}))$.

As previously, if $\varphi'(b'_{r'}) \in K_2$, or $p' = 1$ we have $\varphi'(b_{r'}) = d_{i+1}$ and $j = n_i$. Otherwise, if d_i^{j+1} denotes $\alpha(\sigma^{p'-1}(\alpha(b'_{r'})))$ we have $CW(d_i^{j+1}) - \{d_i^{j+1}\} \subset \varphi'^*(d)$ and P_i can be written as:

$$P_i = CW(d_i) \ldots (CW(d_i^j) - \{d_i^j\})(CW(d_i^{j+1}) - \{d_i^{j+1}\}) \ldots$$

Thus the recursive hypothesis holds until $j+1$.

□

Intuitively, this last lemma may be interpreted has follow: Since each connecting walk is included in one face of the initial combinatorial map, each face of the initial combinatorial map may be considered has a concatenation of connecting walks. Using the restricted combinatorial map G', we have to remove the darts which belong to $\boldsymbol{\mathcal{D}} - K_2 \cup K_1 = \mathcal{SD}_1 - K_2$. The removed dart being surviving ones, we only have to remove the starting dart of some connecting walks.

The following lemma show that each P_i in included in one tree of the contraction kernel K_1. Intuitively, this last proposition is true because the walk P_i does not cross a surviving dart (we never have $d \in \mathcal{SD}_1$ and $\varphi(d)$ in the same walk). Since the surviving darts connect the different trees of the contraction kernel a walk P_i must remains in a given tree.

Lemma 7 *Let us use the same notation for the walks $P_1, \ldots, P_m$ and the hypothesis as in Lemma 6. If, the dart d belongs to K_2 then every walk P_i*

consists either of the dart d_i alone or all the other darts of P_i are part of one connected component of K_1.

$$\forall i \in \{1,\dots,m\} \quad P_i = (d_i) \text{ or } \quad \exists! \mathcal{T} \in \mathcal{CC}(K_1) \mid P_i - \{d_i\} \subset \mathcal{T}$$

Proof:

Let us consider a given walk P_i with $i \in \{1,\dots,m\}$ such that $P_i \neq (d_i)$, we have:

$$P_i = CW(d_i).(CW(d_i^1) - (d_i^1)) \dots (CW(d_i^{n_i}) - (d_i^{n_i}))$$

Since $P_i \neq (d_i)$, $CW(d_i)$ is not reduced to d_i and we have by proposition 4:

$$\exists! \mathcal{T} \in \mathcal{CC}(K_1) \mid CW(d_i) - \{d_i\} \subset \mathcal{T}$$

If P_i is reduced to $CW(d_i)$ nothing remains to be demonstrated. Otherwise, let:

$$CW(d_i) = b_1, \dots, b_r$$

Since $P_i = CW(d_i).(CW(d_i^1) - \{d_i^1\}) \dots \subset \varphi'^*(d)$, we have $\varphi'(b_r) \in CW(d_i^1) - \{d_i^1\}$ with (see Figure 5):

$$\varphi'(b_r) = \sigma^p(\alpha(b_r)) \text{ with } p = Min\{k \in \mathbb{N}^* \mid \sigma^k(\alpha(b_r)) \in K_2 \cup K_1\}$$

Therefore, $\varphi'(b_r) \in \sigma^*(\alpha(b_r))$ with $b_r \in \mathcal{T}$. By definition of a contraction kernel, we have:

$$\sigma^*(\alpha(b_r)) \subset \mathcal{T} \cup \mathcal{SD}_1$$

Since $P_i \neq CW(d_i)$, $\varphi'(b_r)$ is not a surviving dart, thus we have $\varphi'(b_r) \in \mathcal{T}$. Therefore:

$$CW(d_i^1) - \{d_i^1\} \subset \mathcal{T}$$

Let us suppose that this property is true until the rank k with $k < n_i$. Then, if:

$$CW(d_i^k) = d_i^k \dots, b'_{r'}$$

We have $\varphi'(b'_{r'}) \in CW(d_i^{k+1}) - \{d_i^{k+1}\}$ with $b'_{r'}$ belonging to $\mathcal{T}$ by our recurrence hypothesis. We can thus conclude has previously that:

$$CW(d_i^{k+1}) - \{d_i^{k+1}\} \subset \mathcal{T}$$

This property being true for all k in $\{1,\dots,n_i\}$, we have $P_i \subset \mathcal{T}$. $\square$

Lemma 8 *Let us use the same notation for the darts $d_1, \ldots, d_m$ and the hypothesis as in Lemmata 6 and 7. If d belongs to K_2 the ordered set of darts $d_1, \ldots, d_m$ satisfy the following relationship:*

$$\forall i \in \{1, \ldots, m\} \quad d_{i+1} = \varphi'_1(d_i)$$

Where φ'_1 denotes the permutation φ of the sub map G'_1 of G_1(see Figure 3).

Proof:

Using Lemma 6, the set of darts between two consecutive darts d_i and d_{i+1} may be decomposed into the set of connecting walks:

$$\begin{array}{l} P_i = CW(d_i).(CW(d_i^1) - (d_i^1)) \ldots (CW(d_i^{n_i}) - (d_i^{n_i})) \text{ with} \\ \forall j \in \{1, \ldots, n_i\} \quad d_i^j \in \mathcal{SD}_1 - K_2 \end{array}$$

Let us first show that, for a given i in $\{1, \ldots, m\}$, we have:

$$\begin{array}{l} \forall j \in \{1, \ldots, n_i - 1\} \quad \exists k_j \mid \alpha(d_i^{j+1}) = \sigma_1^{k_j}(\alpha(d_i^j)) \text{ with} \\ \forall k \in \{1, \ldots, k_j\} \quad \sigma_1^k(\alpha(d_i^j)) \in \mathcal{SD}_1 - K_2 \end{array} \tag{4}$$

Let us consider a given j in $\{1, \ldots, n_i - 1\}$ and let us denote $CW(d_i^j)$ by:

$$CW(d_i^j) = d_i^j, b_1, \ldots, b_r$$

we have:

$$\varphi_1(d_i^j) = follow(d_i^j) = \varphi(b_r) = \sigma(\alpha(b_r))$$

Moreover, by definition of a submap it exists one k_j (see Figure 6) such that $\varphi'(b_r) = \sigma^{k_j+1}(\alpha(b_r))$ with:

$$\forall k \in \{1, \ldots, k_j\} \quad \sigma^k(\alpha(b_r)) \in \boldsymbol{\mathcal{D}} - K_2 \cup K_1 = \mathcal{SD}_1 - K_2$$

If $k_j = 0$, we have $\varphi'(b_r) = \sigma(\alpha(b_r)) = \varphi_1(d_i^j) \in \mathcal{SD}_1 \cap K_2 \cup K_1 = K_2$ (see Lemma 4). Thus $d_{i+1} = \varphi_1(d_i^j)$. This last equality is in contradiction with our hypothesis: $j < n_i$. Thus we have $k_j \geq 1$. Moreover, since $\sigma^k(\alpha(b_r))$ belongs to $\mathcal{SD}_1$ for each k in $\{1, \ldots, k_j\}$ we have:

$$\forall k \in \{1, \ldots, k_j - 1\} \quad CW(\alpha(\sigma^k(\alpha(b_r)))) = \alpha(\sigma^k(\alpha(b_r)))$$

Thus :

$$\forall k \in \{1, \ldots, k_j - 1\} \quad \left\{ \begin{array}{lcl} \sigma_1(\sigma^k(\alpha(b_r))) & = & follow(\alpha(\sigma^k(\alpha(b_r)))) \\ & = & \varphi(\alpha(\sigma^k(\alpha(b_r)))) \\ & = & \sigma^{k+1}(\alpha(b_r)) \end{array} \right.$$

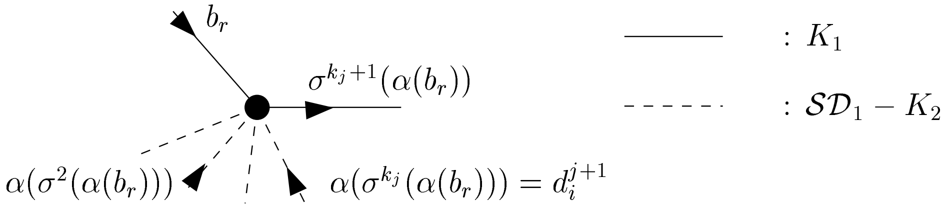

Figure 6: *A zoom on a connection between two connecting walks of a φ'-orbit. In this example, $k_j = 4$*

This last equality may be iterated in order to obtain:

$$\sigma_1^{k_j-1}(\sigma(\alpha(b_r))) = \sigma_1^{k_j-2}(\sigma^2(\alpha(b_r))) = \ldots = \sigma^{k_j}(\alpha(b_r))$$

Therefore:

$$\begin{array}{rcl} \sigma_1^{k_j}(\alpha(d_i^j)) & = & \sigma_1^{k_j-1}(\sigma_1(\alpha(d_i^j))) \\ & = & \sigma_1^{k_j-1}(\sigma(\alpha(b_r))) \\ & = & \sigma^{k_j}(\alpha(b_r)) \\ & = & \alpha \circ \varphi^{-1} \circ \sigma^{k_j+1}(\alpha(b_r)) \\ & = & \alpha(d_i^{j+1}) \end{array}$$

We have thus:

$$\alpha(d_i^{j+1}) = \sigma_1^{k_j}(\alpha(d_i^j)) \text{ with } \forall k \in \{1, \ldots, k_j\} \quad \sigma_1^k(\alpha(d_i^j)) \in \mathcal{SD}_1 - K_2$$

In the same way, if b' denotes the last dart of $CW(d_i^{n_i})$ we have:

$$\begin{array}{lclcl} \varphi_1(d_i^{n_i}) & = & follow(d_i^{n_i}) & = & \sigma(\alpha(b')) \\ d_{i+1} & = & \varphi'(b') & = & \sigma^p(\alpha(b')) \end{array}$$

If $p = 1$, we have $\varphi'(b') = \sigma(\alpha(b')) = \varphi_1(d_i^{n_i}) = \sigma_1(\alpha(d_i^{n_i}))$, thus $d_{i+1} = \sigma_1(\alpha(d_i^{n_i}))$ and we can take $k_{n_i} = 1$. Otherwise, the same demonstration as above may be applied with $k_{n_i} = p - 1$. We thus obtain:

$$d_{i+1} = \sigma_1^{k_{n_i}}(\alpha(d_i^{n_i})) \text{ with } \forall k \in \{1, \ldots, k_{n_i} - 1\} \quad \sigma_1^k(\alpha(d_i^{n_i})) \in \mathcal{SD}_1 - K_2$$

Using equation 4 we have:

$$d_{i+1} = \sigma_1^{k_{n_i}} \circ \sigma_1^{k_{n_i-1}} \ldots \circ \sigma_1^{k_1}(\alpha(d_i)) = \sigma_1^q(\alpha(d_i))$$

with $q = \sum_{j=1}^{n_i} k_j$. Moreover, we have:

$$\forall k \in \{1, \ldots, q\} \quad \sigma_1^k(\alpha(d_i)) \in \mathcal{SD}_1 - K_2$$

Thus: $\varphi_1'(\alpha(d_i)) = \sigma_1'(\alpha(d_i)) = \sigma_1^q(\alpha(d_i))$. $\square$

The Lemma 8 shows that the φ-orbits of combinatorial map G_1' are included in the φ-orbits of the combinatorial map G'. Thus, the combinatorial map G_1' may be considered as the dual of the restriction to K_2 of the combinatorial map $\overline{G'}$:

$$G_1' = \overline{\overline{G'}_{|K_2}} \tag{5}$$

where $\overline{G'}_{|K_2}$ denotes the subgraph of $\overline{G'}$ defined by K_2.

Theorem 4 *Given a combinatorial map $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, and two contraction kernels K_1 and K_2 such that $K_1 \prec K_2$. Then $K_1 \cup K_2$ defines a new contraction kernel such that: $(G_0/K_1)/K_2 = G_0/(K_1 \cup K_2)$.*

Proof:

Given the combinatorial map $G' = (K_2 \cup K_1, \sigma', \alpha)$ defined as the restriction of G_0 to $K_2 \cup K_1$, let us suppose that G' is not a forest, thus that we can find one dart d in $K_2 \cup K_1$ such that $\alpha(d) \notin \varphi'^*(d)$.

The set K_1 being a contraction kernel, and thus a forest, $\varphi'^*(d)$ cannot be included in K_1. Thus:

$$\varphi'^*(d) \cap K_2 \neq \emptyset$$

Let us denote by $\{d_1, d_2, \ldots, d_m\}$ the previous intersection:

$$\{d_1, d_2, \ldots, d_m\} = \varphi'^*(d) \cap K_2$$

Using Lemma 8,the darts $d_1, \ldots, d_m$ define a φ_1'-orbit:

$$\varphi_1'^*(d_1) = (d_1, d_2, \ldots, d_m)$$

In the same way, we can define a set of darts $\{d_1', \ldots, d_{m'}'\}$ such that:

$$\{d_1', d_2', \ldots, d_{m'}'\} = \varphi'^*(\alpha(d)) \cap K_2$$

Using Lemma 8, we obtain:

$$\varphi_1'^*(d_1') = (d_1', d_2', \ldots, d_{m'}')$$

Since two orbits of a permutation are equal or disjoint and $\alpha(d) \notin \varphi'^*(d)$ the orbits $\varphi'^*(d)$ and $\varphi'^*(\alpha(d))$ must be disjoint. Thus,

$$\{d_1, \ldots, d_m\} \cap \{d'_1, \ldots, d'_{m'}\} = \emptyset$$

Moreover, if d belongs to K_2, it exists two indices i and j such that $d = d_i$ and $\alpha(d) = d'_j$. Thus, in this case, the two faces, $\varphi'^*(d_i)$ and $\varphi'^*(d'_j)$ belong to a same connected component.

Otherwise, since d belongs to $K_2 \cup K_1$, $\alpha^*(d)$ is included in K_1. By definition of a contraction kernel, it exists a tree $\mathcal{T} \in \mathcal{CC}(K_1)$ such that $\alpha^*(d) \subset \mathcal{T}$. Moreover, using Lemma 6 we can find two walks P_i and P'_j such that:

$$\begin{cases} d & \in & P_i & \subset & \varphi'^*(d) \\ \alpha(d) & \in & P'_j & \subset & \varphi'^*(\alpha(d)) \end{cases}$$

Using Lemma 7, the non-surviving darts of each walk belong to one tree of K_1. Since $\alpha^*(d) \subset \mathcal{T}$ the walks $P_i - \{d_i\}$ and $P'_j - \{d'_j\}$ are included in $\mathcal{T}$:

$$\begin{array}{lllll} d & \in & P_i - \{d_i\} & \subset & \mathcal{T} \text{ and} \\ \alpha(d) & \in & P'_j - \{d'_j\} & \subset & \mathcal{T} \end{array}$$

Moreover:

$$\left.\begin{array}{lll} \varphi(d_i) & = & \sigma(\alpha(d_i)) \in \mathcal{T} \\ & \text{and} & \\ \varphi(d'_j) & = & \sigma(\alpha(d'_j)) \in \mathcal{T} \end{array}\right) \Rightarrow \alpha(\{d_i, d'_j\}) \subset \sigma^*(\mathcal{T}) \cap \mathcal{SD}.$$

Using proposition 10 we obtain: $\alpha(d'_j) \in \sigma_1^*(\alpha(d_i))$, Thus $\alpha(d'_j) \in \sigma_1'^*(\alpha(d_i))$.

Thus in all cases ($\alpha^*(d) \subset K_2$ or $\alpha^*(d) \subset K_1$) the two faces $\varphi_1'^*(d_1)$ and $\varphi_1'^*(d'_1)$ belong to the same connected component of the submap G'_1 of G_1.

Thus we can find two distinct faces in one connected component of G'_1. The map G'_1 being a forest, by definition of the contraction kernel K_2, we obtain the desired contradiction. Therefore, each connected component of G' is a tree and G' is a forest. Moreover, we have, by lemma 4:

$$\boldsymbol{\mathcal{D}} - K_2 \cup K_1 = \mathcal{SD}_1 - K_2 \neq \emptyset$$

Thus, the connected components of G' define a contraction kernel on G_0. $\square$

3.2 Deriving Successor Kernels from Inclusion Kernels

We will show in this section, that given two contraction kernels K_1 and K_2 such that $K_1 \subset K_2$ we can find another contraction kernel K_2' such that the successive applications of K_1 and K_2' is equivalent to the application of K_2.

Proposition 11 *Given a combinatorial map G_0 and two contraction kernels $K_1 \subset K_2$. A tree $\mathcal{T}$ of K_1 cannot be adjacent to a tree $\mathcal{T}'$ of K_2 unless $\mathcal{T} \subset \mathcal{T}'$:*

$$\forall \mathcal{T} \in \mathcal{CC}(K_1), \forall \mathcal{T}' \in \mathcal{CC}(K_2) \quad \sigma^*(\mathcal{T}) \cap \sigma^*(\mathcal{T}') \neq \emptyset \Rightarrow \mathcal{T} \subset \mathcal{T}'$$

Proof:

The set $\mathcal{T}$ is included in one $\mathcal{T}''$. Let us suppose that $\mathcal{T}' \neq \mathcal{T}''$, we have then:

$$\sigma^*(\mathcal{T}) \cap \sigma^*(\mathcal{T}') \subset \sigma^*(\mathcal{T}'') \cap \sigma^*(\mathcal{T}')$$

By definition of a contraction kernel we must have, since $\mathcal{T}' \neq \mathcal{T}''$

$$\sigma^*(\mathcal{T}') \cap \sigma^*(\mathcal{T}'') = \emptyset$$

This last equation contradict the hypothesis $\sigma^*(\mathcal{T}) \cap \sigma^*(\mathcal{T}') \neq \emptyset$. $\square$

Proposition 12 *Given a combinatorial map G_0 and two contraction kernels K_1 and K_2 with $K_1 \subset K_2$. If G_1 and G_2 denote the two contracted maps:*

$$\begin{array}{rclcl} G_1 & = & (\mathcal{SD}_1, \sigma_1, \alpha) & = & G/K_1 \\ G_2 & = & (\mathcal{SD}_2, \sigma_2, \alpha) & = & G/K_2 \end{array}$$

then the smaller contraction kernel K_1 creates the larger graph:

$$\mathcal{SD}_2 \subset \mathcal{SD}_1$$

where $\mathcal{SD}_1$ and $\mathcal{SD}_2$ denote respectively the surviving darts of K_1 and K_2.

Proof:

The contraction kernel K_1 is included in K_2, therefore $\mathcal{SD}_2 = \boldsymbol{\mathcal{D}} - K_2$ is included in $\mathcal{SD}_1 = \boldsymbol{\mathcal{D}} - K_1$. $\square$

Proposition 13 *Given a combinatorial map G_0 and two contraction kernels K_1 and K_2 with $K_1 \subset K_2$. Each tree $\mathcal{T}' \in \mathcal{CC}(K_2)$ may be written as an union of trees $\mathcal{T} \in \mathcal{CC}(K_1)$ together with some surviving darts in $\mathcal{SD}_1$ which connect the trees of K_1 included in $\mathcal{T}'$:*

$$\begin{array}{l} \forall \mathcal{T}' \in \mathcal{CC}(K_2) \quad \exists C_{\mathcal{T}'} \subset \mathcal{CC}(K_1) \quad | \quad \mathcal{T}' = (\cup_{\mathcal{T} \in C_{\mathcal{T}'}} \mathcal{T}) \cup A_{\mathcal{T}'} \\ \text{with } A_{\mathcal{T}'} \subset \mathcal{SD}_1 \end{array}$$

where $\mathcal{SD}_1$ denotes the set of surviving darts of the contraction kernel K_1.

Proof:

Given the set:

$$C_{\mathcal{T}'} = \{\mathcal{T} \in \mathcal{CC}(K_1) \quad | \quad \mathcal{T} \subset \mathcal{T}'\}$$

We have only to prove that:

$$\forall \mathcal{T}' \in \mathcal{CC}(K_2) \quad A_{\mathcal{T}'} = \mathcal{T}' - \bigcup_{\mathcal{T} \in C_{\mathcal{T}'}} \mathcal{T} \subset \mathcal{SD}_1$$

Let us consider $d \in \mathcal{T}'$, then $d \in \mathcal{SD}_1$ or belongs to one $\mathcal{T} \in \mathcal{CC}(K_1)$. Let us suppose that $d \in \mathcal{T}$ with $\mathcal{T} \notin C_{\mathcal{T}'}$. Then, by definition of included contraction kernels there must exist $\mathcal{T}'' \neq \mathcal{T}'$ such that: $d \in \mathcal{T} \subset \mathcal{T}''$. Thus $d \in \mathcal{T}' \cap \mathcal{T}''$ which is forbidden by the definition of a contraction kernel. Thus we must have $\mathcal{T} \in C_{\mathcal{T}'}$. Therefore, if $d \in \mathcal{T}' - \cup_{C_{\mathcal{T}'}} \mathcal{T}$ then $d \in \mathcal{SD}_1$. $\square$

Lemma 9 *Using the same notations and hypothesis as proposition 13, the sets $C_{\mathcal{T}'}$ with $\mathcal{T}' \in \mathcal{CC}(K_2)$ form a partition of $\mathcal{CC}(K_1)$:*

$$\bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} C_{\mathcal{T}'} = \mathcal{CC}(K_1)$$

Proof:

If a tree $\mathcal{T}$ of $\mathcal{CC}(K_1)$ belongs to two sets $C_{\mathcal{T}'}$ and $C_{\mathcal{T}''}$ with $\mathcal{T}' \neq \mathcal{T}''$, we have $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T} \subset \mathcal{T}''$. Therefore, $\mathcal{T}$ connects $\mathcal{T}'$ to $\mathcal{T}''$. This is in contradiction with the definition of the trees $\mathcal{T}'$ and $\mathcal{T}''$ as connected components of K_2. Therefore, all the sets $C_{\mathcal{T}'}$ with $\mathcal{T}'$ in $\mathcal{CC}(K_2)$ are disjoint. Let us show that these sets form a partition of $\mathcal{CC}(K_1)$.

For every $\mathcal{T}' \in \mathcal{CC}(K_2)$, $C_{\mathcal{T}'}$ is included in $\mathcal{CC}(K_1)$, thus:

$$\bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} C_{\mathcal{T}'} \subset \mathcal{CC}(K_1)$$

If $\mathcal{T} \in \mathcal{CC}(K_1)$, we have, by definition of included contraction kernels:

$$\exists! \mathcal{T}' \in \mathcal{CC}(K_2) \quad | \quad \mathcal{T} \subset \mathcal{T}'$$

Thus: $\mathcal{T} \in C_{\mathcal{T}'} \subset \bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} C_{\mathcal{T}'}$.
Finally we obtain:

$$\bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} C_{\mathcal{T}'} = \mathcal{CC}(K_1)$$

$\square$

Corollary 4 *Using the same notations and hypothesis as proposition 13, the sets $A_{\mathcal{T}'}$ with $\mathcal{T}' \in \mathcal{CC}(K_2)$ form a partition of $K_2 - K_1$:*

$$\bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} A_{\mathcal{T}'} = K_2 - K_1$$

Proof:
Given a tree $\mathcal{T}'$ in $\mathcal{CC}(K_2)$, $A_{\mathcal{T}'}$ is equal to (see proposition 13):

$$A_{\mathcal{T}'} = \mathcal{T}' - \bigcup_{\mathcal{T} \in C_{\mathcal{T}'}} \mathcal{T}$$

Each set $A_{\mathcal{T}'}$ is included in $\mathcal{T}'$. The connected component of K_2 being disjoint by definition, the sets $A_{\mathcal{T}'}$ are disjoint. The union of all the sets $A_{\mathcal{T}'}$ is thus equal to:

$$\begin{aligned} \bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} A_{\mathcal{T}'} &= \bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} \mathcal{T}' - \bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} \bigcup_{\mathcal{T} \in C_{\mathcal{T}'}} \mathcal{T} \\ &= K_2 - \bigcup_{\mathcal{T} \in \mathcal{CC}(K_1)} \mathcal{T} \text{ (with lemma 9)} \\ &= K_2 - K_1 \end{aligned}$$

$\square$

Lemma 10 *With the same notations and hypothesis as proposition 13, if* $K_2 - K_1$ *is not empty, it defines a forest of* G_0.

Proof:

The contraction kernels K_2 and K_1 being symmetric, the set $K_2 - K_1$ is symmetric and can be considered as a sub-combinatorial map of the forest K_2. We can thus conclude with Theorem 1. $\square$

This last lemma will be used by Theorem 5 to contract G_0.

Theorem 5 *Given a combinatorial map* G_0, *and two contraction kernels* K_1 *and* K_2. *If:*

$$\begin{array}{rcl} G_1 & = & G_0/K_1 \\ G_2 & = & G_0/K_2 \end{array}$$

and if K_1 *is included in* K_2, *e.g.* $K_1 \subset K_2$, G_2 *can be derived from* G_1 *by additional contractions using the same notations as in Proposition 13:*

$$G_2 = G_1/(K_2 - K_1)$$

Proof:

$$\begin{array}{rcrcl} G_2 & = & G_0 & / & K_2 \\ & = & G_0 & / & (K_1 \cup (K_2 - K_1)) \\ & = & (G_0 & / & K_1)/(K_2 - K_1) \text{ (with lemma 10 and 3)} \\ & = & G_1 & / & (K_2 - K_1) \end{array}$$

$\square$

We know, thanks to Theorem 5, that contracting the darts $K_2 - K_1$ on the combinatorial map G_1 provides the combinatorial map G_2. We have now to show that a contraction kernel of G_1 may be defined on the set $K_2 - K_1$. This last hypothesis may be easily shown if we demonstrate that $K_2 - K_1$ is a forest of the combinatorial map G_1. The following lemma, defines an intermediate result,which is used in the proof of proposition 14.

Lemma 11 *Given a combinatorial map* $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, *two contraction kernels* K_1 *and* K_2 *with* $K_1 \subset K_2$ *and the contracted graph:*

$$G_1 = (\mathcal{SD}_1, \sigma_1, \alpha) = G_0/K_1$$

Using the same notations as proposition 13, the σ_1-orbits of each set $A_{\mathcal{T}'}$ is included in the σ-orbits of $\mathcal{T}'$ intersected with $\mathcal{SD}_1$:

$$\forall \mathcal{T}' \in \mathcal{CC}(K_2) \quad \sigma_1^*(A_{\mathcal{T}'}) \subset \sigma^*(\mathcal{T}') \cap \mathcal{SD}_1$$

Proof:

First note that $\sigma_1^*(A_{\mathcal{T}'})$ is well defined since $A_{\mathcal{T}'}$ is included in $\mathcal{SD}_1$.

Let us consider d in the σ_1 orbit of $A_{\mathcal{T}'}$. In this case the σ_1-orbit of d intersect $A_{\mathcal{T}'}$:

$$\sigma_1^*(d) \cap A_{\mathcal{T}'} \neq \emptyset$$

Let us consider the two cases:

1. If $\sigma^*(d)$ is included in $\mathcal{SD}_1$. Then we have thanks to corollary 1 and to the isomorphism between the contracted graph G_1 and the connecting walks map $G_{K_1} = (\boldsymbol{\mathcal{D}}_{K_1}, \sigma_{K_1}, \alpha_{K_1})$:

$$\sigma_1^*(d) = CW^{-1}(\sigma_{K_1}^*(CW(d)) = CW^{-1}(CW(\sigma^*(d))) = \sigma^*(d).$$

   Thus:

$$\emptyset \neq \sigma_1^*(d) \cap A_{\mathcal{T}'} \subset \sigma_1^*(d) \cap \mathcal{T}' = \sigma^*(d) \cap \mathcal{T}' \Rightarrow d \in \sigma^*(\mathcal{T}')$$

   Moreover, d belongs to $\mathcal{SD}_1$ by definition of G_1. Thus:

$$d \in \sigma^*(\mathcal{T}') \cap \mathcal{SD}_1$$

2. If $\sigma^*(d)$ intersects one tree $\mathcal{T} \in \mathcal{CC}(K_1)$, we have thanks to proposition 10:

$$\sigma_1^*(d) = CW^{-1}(\sigma_{K_1}^*(CW(d))) = \sigma^*(\mathcal{T}) \cap \mathcal{SD}_1 \subset \sigma^*(\mathcal{T})$$

   Thus:

$$\emptyset \neq \sigma_1^*(d) \cap \mathcal{T}' \subset \sigma^*(\mathcal{T}) \cap \mathcal{T}'$$

   Using proposition 11 we can deduce:

$$\mathcal{T} \subset \mathcal{T}'$$

   Thus:

$$d \in \sigma_1^*(d) = \sigma^*(\mathcal{T}) \cap \mathcal{SD}_1 \subset \sigma^*(\mathcal{T}') \cap \mathcal{SD}_1$$

$\square$

Corollary 5 *The intersection between the σ_1-orbits of any two sets $A_{\mathcal{T}'}$ and $A_{\mathcal{T}''}$ is empty:*

$$\forall(\mathcal{T}', \mathcal{T}'') \in \mathcal{CC}(K_2)^2 \quad \sigma_1^*(A_{\mathcal{T}'}) \cap \sigma_1^*(A_{\mathcal{T}''}) = \emptyset$$

Proof:

Given two distinct trees $\mathcal{T}'$ and $\mathcal{T}''$ in $\mathcal{CC}(K_2)$, the σ_1-orbits of the sets $A_{\mathcal{T}'}$ and $A_{\mathcal{T}''}$ are included in the σ-orbits of $\mathcal{T}'$ and $\mathcal{T}''$ (see lemma 11) which are disjoint:

$$\sigma_1^*(A_{\mathcal{T}'}) \cap \sigma_1^*(A_{\mathcal{T}''}) \subset \sigma^*(\mathcal{T}') \cap \sigma^*(\mathcal{T}'') = \emptyset$$

$\square$

Lemma 12 *Given a combinatorial map $G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha)$, two contraction kernels K_1 and K_2 with $K_1 \subset K_2$ and the two contracted graphs:*

$$\begin{array}{rcccl} G_1 & = & (\mathcal{SD}_1, \sigma_1, \alpha) & = & G_0/K_1 \\ G_2 & = & (\mathcal{SD}_2, \sigma_2, \alpha) & = & G_0/K_2 \end{array}$$

Any walk of G_1 included in $K_2 - K_1$ is included in a given $A_{\mathcal{T}'}$ with $\mathcal{T}' \in \mathcal{CC}(K_2)$.

Proof:

First let us note that (see corollary 4):

$$K_2 - K_1 = \bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} A_{\mathcal{T}'}$$

Let us consider a walk $W = (d_1, \ldots, d_n)$ included in $K_2 - K_1$. Since the sets $A_{\mathcal{T}'}$ form a partition of $K_2 - K_1$ it exists a given tree $\mathcal{T}'$ such that $d_1 \in A_{\mathcal{T}'}$.

Let us consider, the length of the longest sequence of W starting from d_1 and included in $A_{\mathcal{T}'}$:

$$r = Max\{s \in \{1, \ldots, n\} \quad | \quad \forall i \in \{1, \ldots, s\} \quad d_i \in A_{\mathcal{T}'}\}$$

We have $r \geq 0$, let us suppose that $r < n$, then we have: $d_r \in A_{\mathcal{T}'}$ and $d_{r+1} \in A_{\mathcal{T}''}$ with $\mathcal{T}'' \neq \mathcal{T}'$. Moreover, we have, by definition of a walk:

$$d_{r+1} \in \sigma_1^*(\alpha(d_r)) \subset \sigma_1^*(A_{\mathcal{T}'})$$

Thus:

$$d_{r+1} \in A_{\mathcal{T}''} \cap \sigma_1^*(A_{\mathcal{T}'}) \subset \sigma_1^*(A_{\mathcal{T}''}) \cap \sigma_1^*(A_{\mathcal{T}'})$$

this is in contradiction with corollary 5. $\square$

Intuitively, this last proposition may be understood has follow: The set $A_{\mathcal{T}'}$ only connects the different sets $(\mathcal{T})_{\mathcal{T} \in C(\mathcal{T}')}$ included in $\mathcal{T}'$ and not $\mathcal{T}'$ to another tree $\mathcal{T}''$. Thus a walk defined in $K_2 - K_1 = \cup_{\mathcal{T}' \in \mathcal{CC}(K_2)} A_{\mathcal{T}'}$ cannot connect two trees $\mathcal{T}'$ and $\mathcal{T}''$ and is therefore included in one $A_{\mathcal{T}'}$.

Proposition 14 *Given a combinatorial map G_0, two contraction kernels $K_1 \subset K_2$ and the two contracted graphs:*

$$\begin{array}{rcccl} G_1 & = & (\mathcal{SD}_1, \sigma_1, \alpha) & = & G_0/K_1 \\ G_2 & = & (\mathcal{SD}_2, \sigma_2, \alpha) & = & G_0/K_2 \end{array}$$

The submap K_{01} of G_1 defined by $K_2 - K_1$ is a forest (see Figure 7 for an illustration of the relationships between the different combinatorial maps).

$$\begin{tikzcd}
 & & G_0 = (\boldsymbol{\mathcal{D}}, \sigma, \alpha) \arrow[d, "K_1"] \arrow[dd, bend left=60, "K_2"] \\
K_{01} = (K_2 - K_1, \sigma_1', \alpha) & \subset & G_1 = (\mathcal{SD}_1 = \boldsymbol{\mathcal{D}} - K_1, \sigma_1, \alpha) \\
 & & G_2 = (\mathcal{SD}_2 = \boldsymbol{\mathcal{D}} - K_2, \sigma_2, \alpha)
\end{tikzcd}$$

Figure 7: *The relationships between the combinatorial maps G_0, G_1, G_2 and $K_{01} = (K_2 - K_1, \sigma_1', \alpha)$. The arrows represent contractions.*

Proof:

We will demonstrate this important proposition by showing that if K_{01} is not a forest, then we can find a cycle of G_0 included in one tree $\mathcal{T}'$ of $\mathcal{CC}(K_2)$.

If K_{01} is not a forest of G_1, then we can find a cycle C of G_1 included in K_{01}. A cycle, being also a walk, we have by lemma 12:

$$\exists \mathcal{T}' \in \mathcal{CC}(K_2) \quad | \quad C \subset A_{\mathcal{T}'} \subset \mathcal{T}'$$

if $C_{\mathcal{T}'}$ is empty, the set $A_{\mathcal{T}'}$ defines a new tree:

$$C \subset A_{\mathcal{T}'} = \mathcal{T}' \subset \mathcal{SD}_1$$

The permutation σ_1 and σ being identical on $\mathcal{SD}_1$ (see proposition 9 and theorem 3) C is also a cycle of G_0. We thus obtain the desired contradiction since C is included in $\mathcal{T}'$ which is a tree of G_0 (see theorem 2).

If $C_{\mathcal{T}'} \neq \emptyset$, let us show that we can extend C into an other cycle C' of G_0 included in $\mathcal{T}'$. If C is defined by the darts $d_1, \ldots, d_n$, let us write C' (see Figure 8) as:

$$C' = d_1.P_1 \ldots d_n P_n$$

Where $(P_i)_{i\in\{1,\ldots,n\}}$ denotes a set of path to be determined.

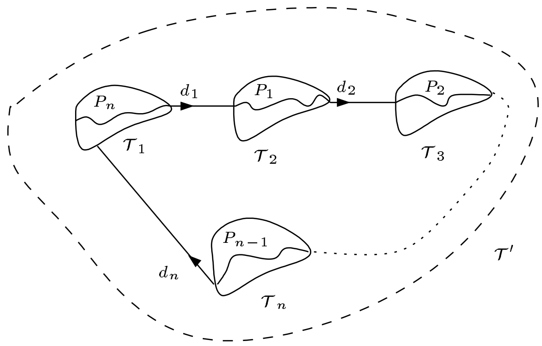

Figure 8: *The cycle $C = (d_1, \ldots, d_n)$ extended to $C' = (d_1.P_1 \ldots d_n P_n)$*

Given an index i in $\{1, \ldots, n\}$, Let us consider two cases:

1. If $\sigma^*(\alpha(d_i)) \subset \mathcal{SD}_1$, the σ_1-orbit of $\alpha(d_i)$ is equal to its σ-orbit, and we have:

   $$d_{i+1} \in \sigma_1^*(\alpha(d_i)) = \sigma^*(\alpha(d_i))$$

   In this case we take $P_i = \emptyset$.

2. If $\alpha(d_i)$ belongs to the σ-orbit of a tree $\mathcal{T}_{i+1} \subset K_1$

   $$d_{i+1} \in \sigma_1^*(\alpha(d_i)) = \sigma^*(\mathcal{T}_{i+1}) \cap \mathcal{SD}_1$$

   Then $\sigma^*(\alpha(d_i))$ and $\sigma^*(d_{i+1})$ belong to the same tree $\mathcal{T}_{i+1}$ of K_1 and it exists a unique path P_i in $\mathcal{T}_{i+1}$ from $\sigma^*(\alpha(d_i))$ to $\sigma^*(d_{i+1}))$ (see theorem 2).

   Note that if $\sigma^*(\alpha(d_i)) = \sigma^*(d_{i+1})$ the path is again empty.

The serie C' so defined is by construction a walk of G_0. Let us show that it is closed.

If $\sigma^*(d_1)$ is included in $\mathcal{SD}_1$, we have:

$$\sigma^*(d_1) \cap \alpha^*(C') = \sigma_1^*(d_1) \cap \alpha^*(C') = \{d_1, \alpha(d_n)\}$$

If not, $\sigma^*(d_1)$ intersects a tree of K_1, and contains d_1 and the opposite of the last dart of P_n by definition of P_n. Therefore the walk C' is closed. Let us show that it is a cycle. If C' is denoted by:

$$C' = b_1, \ldots, b_p$$

we must show that:

$$\forall i \in \{2, \ldots, p\} \quad \sigma^*(b_i) \cap \alpha^*(b_1, \ldots, b_p) = \{b_i, \alpha(b_{i-1})\}$$

If b_i belongs to a path P_j, its σ-orbits contains b_i and $\alpha(b_i)$ by definition of a path. If some other darts in $\alpha^*(C')$ belong to the same σ-orbit, we must have some other darts than d_j and $\alpha(d_{j-1})$ in C incident to the same tree. Since each tree of K_1 is contracted into a single vertex, this is in contradiction with the definition of C as a cycle of G_1.

Given a dart d_i in C, if $\sigma^*(d)$ is included in $\mathcal{SD}_1$, we have:

$$\sigma^*(d_i) \cap \alpha^*(b_1, \ldots, b_p) = \sigma_1^*(d_i) \cap \alpha^*(b_1, \ldots, b_p) = \{d_i, \alpha(d_{i-1})\}$$

If not, $\sigma^*(d_i)$ must contains d_i and the opposite of the last dart of P_{i-1} by definition of the paths P_i. If $\sigma^*(d_i)$ contains some other darts in $\alpha^*(C')$ we contradict as previously the definition of C as a cycle of G_1.

Therefore, C' is a cycle of G_0 included in the tree $\mathcal{T}'$. We obtain the desired contradiction. $\square$

The above demonstration is based on the fact that contractions defined by contraction kernels do not remove nor create cycles. Therefore, a cycle defined in G_0 is contracted in a cycle of G_1. Conversely, a cycle C defined in G_1 can be extended in a cycle C' of G_0 such that the contraction of C' is equal to C.

Using the notations of proposition 14 , the set $K_2 - K_1$ defines a forest K_{01} of G_1. The following theorem shows that $K_2 - K_1$ is also a contraction kernel of G_1.

Theorem 6 *Given a combinatorial map G_0, two contraction kernels $K_1 \subset K_2$. If G_1 denotes the contracted map associated to K_1:*

$$G_1 = (\mathcal{SD}_1, \sigma_1, \alpha) = G/K_1$$

The contraction kernel $K_2 - K_1$ is a successor of K_1.

Proof:

We have only to show that:

$$\mathcal{SD}_1 - (K_2 - K_1) \neq \emptyset$$

We have:

$$\begin{array}{rcl} \mathcal{SD}_1 - (K_2 - K_1) & = & \boldsymbol{\mathcal{D}} - K_1 - (K_2 - K_1) \\ & = & \boldsymbol{\mathcal{D}} - K_2 \\ & = & \mathcal{SD}_2 \neq \emptyset \end{array}$$

Note that this last property may also be demonstrated thanks to proposition 12:

$$(\mathcal{SD}_1 - \bigcup_{\mathcal{T}' \in \mathcal{CC}(K_2)} A_{\mathcal{T}'}) \cap \mathcal{SD}_2 = \mathcal{SD}_2 \cap \mathcal{SD}_1 = \mathcal{SD}_2 \neq \emptyset$$

□

4 Conclusion

We have presented in this technical report the notions of contraction kernel and equivalent contraction kernel. Our main result on equivalent contraction kernels is illustrated in Figure 9 (see also Figure 10): Given the two successor kernels K_1 and K_2, we can define, thanks to Theorem 4, a contraction kernel K_3 with $K_1 \subset K_3$ providing the same combinatorial map than the successive applications of K_1 and K_2 on G_0. Conversely, given two inclusion kernels K_1 and K_3 we can define, thanks to theorem 6, a new contraction kernel K_2 with $K_1 \prec K_2$ such that the successive applications of K_1 and K_2 is equivalent to K_3.

The design of efficient parallel algorithms is under development. The design of such algorithms should be achieved by using some properties of the evolution of connecting walks along contractions. This expected result

$$
\begin{tikzcd}
G_0 \arrow[r, "K_1"] \arrow[dr, "K_3 = K_1 \cup K_2"'] & G_1 \arrow[d, "K_2 = K_3 - K_1"] \\
 & G_2
\end{tikzcd}
$$

Figure 9: *The relations between equivalent contraction kernels*

together with the ones obtain in this report should allow us to study interesting applications of our model such as:segmentation [3, 1, 2, 4], structural matching [17] or integration of moving objects. Finally, the extension of our model to higher dimensional spaces (3D) should be studied.

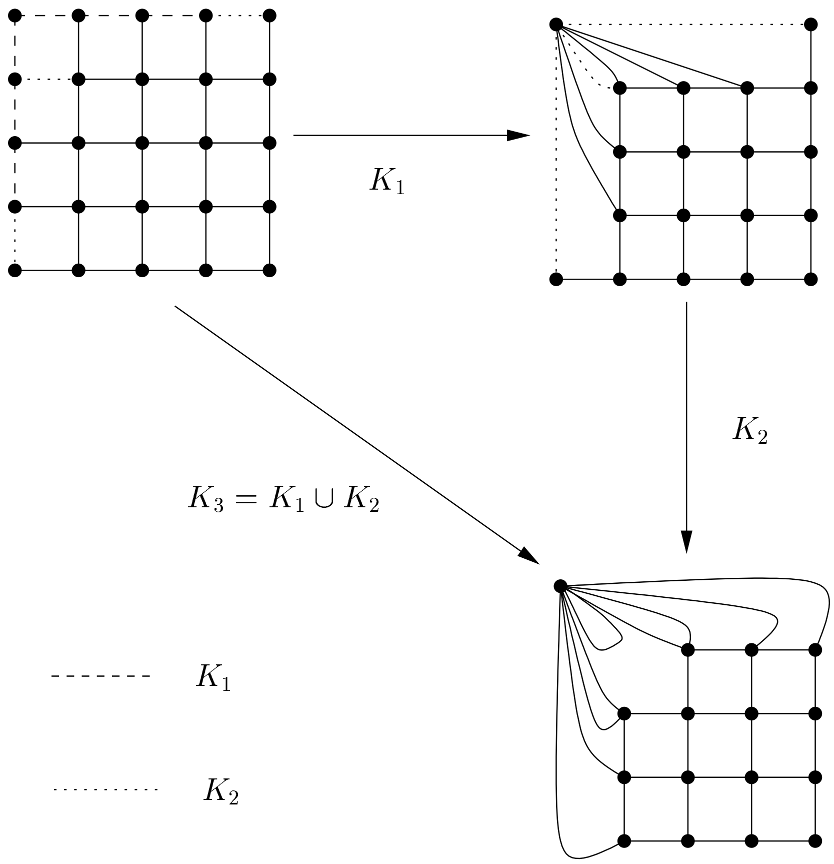

Figure 10: *Two contraction kernels K_1 and K_2 successively applied on a regular grid. The application of $K_2 \circ K_1$ is equivalent to the application of K_3. The contraction kernel K_1 is represented with dashed lines while K_2 is represented with dotted lines.*

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