Decomposition for Efficient Eccentricity Transform of Convex Shapes*

Adrian Ion[1], Samuel Peltier[1], Yll Haxhimusa[1,2], and Walter G. Kropatsch[1]

[1] Pattern Recognition and Image Processing Group,
Faculty of Informatics, Vienna University of Technology, Austria
{ion,sam,yll,krw}@prip.tuwien.ac.at
[2] Department of Psychological Sciences,
Purdue University, USA
yll@psych.purdue.edu

Abstract. The eccentricity transform associates to each point of a shape the shortest distance to the point farthest away from it. It is defined in any dimension, for open and closed manyfolds. Top-down decomposition of the shape can be used to speed up the computation, with some partitions being better suited than others. We study basic convex shapes and their decomposition in the context of the continuous eccentricity transform. We show that these shapes can be decomposed for a more efficient computation. In particular, we provide a study regarding possible decompositions and their properties for the ellipse, the rectangle, and a class of elongated shapes.

1 Introduction

To extract the required information from a set of images, a frequently used pattern is to repeatedly transform the input image while gradually moving from the low abstraction level of the input data to the high abstraction level of the output data. The idea is to have a reduced amount of (important) data at these higher abstraction levels. A class of such transforms applied to 2D shapes, associates to each point of the shape a value that characterizes in some way it's relation to the rest of the shape. This value in many cases is a distance between important points i.e. features.

Examples include the distance transform [1], which associates to each point the length of the **shortest** path to the border, the Poisson equation [2], which can be used to associate to each point the **average** time to reach the border by a random path (an average of the random shortest paths), and the eccentricity transform [3] which associates to each point the **longest** of the shortest paths to any other point of the shape. In this way one tries to come up with an abstracted representation of the shape (e.g. the skeleton [4] or shock graph [5] build on distance transform), which could be easily used in e.g. shape classification or retrieval. Minimal path computation [6] as well as distance transform [7] are used in 2D and 3D image segmentation.

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The eccentricity transform (ECC) has it's origins in the graph based eccentricity [8,9]. Defined in the context of digital images in [3,10], where properties and robustness have been shown, it was applied for shape matching in [11]. The ECC can be computed for of any dimension, discrete closed (e.g. 2D binary image) or open sets (surface of an ellipsoid), and continuous (e.g. 3D ellipsoid or the 2D surface of the 3D ellipsoid, etc.). For discrete 2D shapes, a naive algorithm ($O(N^3)$ in the number of pixels) and a more efficient one for 2D shapes without holes, have been presented in [3].

This paper presents top-down approach for the efficient computation of the ECC of basic convex shapes using decomposition. Section 2 gives a short recall of the ECC. Section 3 shows its computation on an ellipse, rectangle and an elongated shape. Section 4 concludes and gives an outlook of the future work.

2 Recall ECC

In this section basic definitions and properties of the ECC are introduced following [3,11]. Let the shape S be a closed set in $\mathbb{R}^2$ and ∂S be its border[1]. A path π is the continuous mapping from the interval $[0, 1]$ to S. Let $\Pi(p_1, p_2)$ be the set of all paths between two points $p_1, p_2 \in S$ within the set S ($\Pi(p_1, p_2) \subset S$). The geodesic distance $d(p_1, p_2)$ between two points $p_1, p_2 \in S$ is defined as the length λ of the shortest path $\pi(p_1, p_2)$, such that $\pi \in S$, more formally

$$d(p_1, p_2) = \min\{\lambda(\pi(p_1, p_2)) | \pi \in \Pi\} \text{ where } \lambda(\pi(t)) = \int_0^1 |\dot{\pi}(t)| \mathrm{d}t \tag{1}$$

where $\pi(t)$ is a parametrization of the path from $p_1 = \pi(0)$ to $p_2 = \pi(1)$.

The eccentricity transform of S can be defined as, $\forall p \in S$

$$ECC_S(p) = \max\{d(p, q) | q \in S\} = \max\{d(p, q) | q \in \partial S\} \tag{2}$$

i.e. to each point p it assigns the length of the shortest geodesics to the points farthest away from it. In [3] it is shown that this transformation is quasi-invariant to articulated motion and robust against salt and pepper noise [10] (which creates holes in the shape). An *eccentric point* is then the point y that reaches a maximum in Eq. 2. All the eccentric points lie on the border of S [3].

3 ECC of Basic Shapes

In the case of simply connected convex shapes S, geodesic distance equals Euclidean distance, as no obstacles exist that have to be avoided. In the rest of the paper, S denotes a simply connected convex shape, centered at the origin (center of gravity). We will show some properties of shapes S regarding the eccentricity transform, which will help designing new and faster algorithms. For clarity, we introduce the following notations: the set of points $(x, y) \in S$ such that $x > 0$, is denoted S_r and called the *right part* of S, whereas the set of points $(x, y) \in S$ such that $x < 0$, is denoted S_l and called the *left part* of S.

[1] This definition can be generalized to higher dimensions.

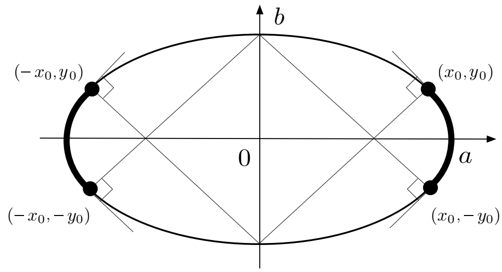

Fig. 1. The sets of eccentric points of the ellipse are shown with a thick line

3.1 Ellipse

Ellipse Recalls. The elliptical curve of points (x, y) around the origin with parameters $a > 0$ and $b > 0$ is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{3}$$

In point (x, y) it has a tangent direction $(\dot{x}, \dot{y})$ satisfying

$$\frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} = 0. \tag{4}$$

Bounding Extremal points. We consider an elliptical region S around the origin, defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. In the following, we assume that $a > b$, thus the *small axis* of S is the segment $[-b, b]$, and the *long axis* of S is the segment $[-a, a]$.

As mentioned before, it has been shown that the eccentric point(s) of any point $(x, y) \in S$, are on the border of S. We now provide two additional properties regarding eccentric points on an elliptical region.

Property 1. Let (x_e, y_e) be an eccentric point for $(x, y) \in S$. Then, the tangent to the ellipse at the point (x_e, y_e) is orthogonal to the line defined by (x, y) and (x_e, y_e).

Proof. Suppose that the tangent is not orthogonal to the line defined by (x, y) and (x_e, y_e), then there exists a point (x'_e, y'_e) in the neighborhood of (x_e, y_e), which is farther away from (x, y) than (x_e, y_e). This would contradict the fact that (x_e, y_e) is an eccentric point for (x, y).

The following property is a general property for symmetric (having one axis of symmetry), simply connected and convex shapes.

Property 2. Let S be a symmetric, simply connected and convex shape, with A its axis of symmetry. Let S_1 and S_2 denote the two symmetric parts of S delineated by A. Any point $p \in S_1$ has its eccentric point p_e in S_2.

Proof. Let $p \in S_1 \backslash A$ and $p_e \in S_1$. Then there exists the symmetric point $p'_e \in S_2$. The straight line connecting p and p'_e intersects A in point $q \in A$ having the same distance to p_e and p'_e: $d(p, p'_e) = d(p, q) + d(q, p'_e) = d(p, q) + d(q, p_e) > d(p, p_e)$ shows that p_e is not eccentric due to the triangular inequality.

The ellipse is a simply connected convex shape and the smaller axis is an axis of symmetry. For any ellipse S, we can choose $S_1 = S_l$ and $S_2 = S_r$.

We compute the eccentric points of $(0, b)$ and $(0, -b)$. This allow us to partition the ellipse into 4 subsegments alternating the property of being eccentric or not (see Fig. 1). Let us consider the line l that goes through the point $(0, -b)$ and crosses the ellipse with orthogonal tangent at point (x_0, y_0), such that $x_0 \geq 0$ and $y_0 \geq 0$. This line $l(\tau)$ is defined by:

$$\begin{cases} x &= \tau \dot{y_0} \\ y &= -b - \tau \dot{x_0}. \end{cases} \tag{5}$$

As $(x_0, y_0) \in l$, we can deduce from Eq. 5 that:

$$\begin{cases} \tau_0 &= \frac{x_0}{\dot{y_0}} \\ y_0 &= -b - \frac{\dot{x_0}}{\dot{y_0}} x_0. \end{cases} \tag{6}$$

From Eq. 4, we obtain $-\frac{\dot{x_0}}{\dot{y_0}} = \frac{y_0 a^2}{x_0 b^2}$. Using Eq. 5, we obtain $y_0 = -b - x_0 \frac{y_0 a^2}{x_0 b^2} = -b - \frac{y_0 a^2}{b^2}$, so

$$y_0 = \frac{b^3}{a^2 - b^2}. \tag{7}$$

The x-coordinate is then determined using the ellipse formula:

$$x_0^2 = a^2 (1 - \frac{b^4}{(a^2 - b^2)^2}). \tag{8}$$

Similar calculations deliver the eccentric point for $(0, -b)$ with $x_0 < 0$ and $y_0 \geq 0$, and the two eccentric points for the point $(0, b)$.

One can directly deduce that any point (x_c, y_c) of the ellipse, s.t. $x_c^2 < x_0^2$ has normals that do not intersect the segment $[-b, b]$. Thus, according to Prop. 2, all these points cannot be eccentric points for any point inside the ellipse. Hence the points $(x_0, y_0), (x_0, -y_0), (-x_0, -y_0), (-x_0, y_0)$ partition the ellipse into 4 subsegments alternating the property of being extremal or not.

Eccentric lines through the smaller axis. In this section, we show how to efficiently compute the eccentricity transform of an elliptical region S, by considering separately S_l and S_r. Using Prop. 1, we first show how to compute the eccentricity of all the points of the small axis.

Let $p = (0, \mu b)$, $-1 \leq \mu \leq 1$ be a point on the small axis, and let $p_e = (x_e, y_e)$ be its eccentric point in S_r. Using Prop. 1, the points (x, y) of the line $l(\tau)$ defined by (p, p_e) satisfy:

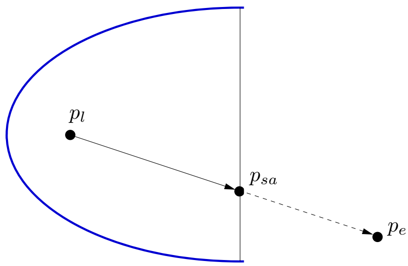

Fig. 2. Efficient computation of eccentricity transform based on decomposition

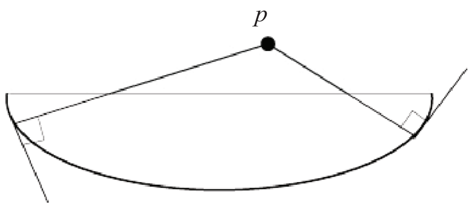

Fig. 3. Ellipse decomposition along the bigger axis: more than one line, orthogonal to the ellipse tangent at the point of intersection, can go through one point

$$\begin{cases} x &= \tau \dot{y_e} \\ y &= \mu b + \tau \dot{x_e}. \end{cases} \tag{9}$$

In particular, $p_e \in l$, so we have $x_e = \tau_e \dot{y_e}$ and $y_e = \mu b + \tau_e \dot{x_e}$. Thus we deduce that $\tau_e = \frac{x_e}{\dot{y_e}}$ and $y_e = \mu b + \frac{\dot{x_e}}{\dot{y_e}} x_e = \frac{\mu b^3}{b^2 - a^2}$.

The x-coordinate is then determined using Eq. 3

$$x_e^2 = a^2(1 - \frac{b^4 \mu^2}{(b^2 - a^2)^2}). \tag{10}$$

So, the eccentricity of any point $(0, \mu b), -1 \leq \mu \leq 1$ of the small axis, can directly be computed by the above formula using a, b and μ. The direction to their eccentric point is also known and can be stored in each point.

As it is a convex shape, the eccentric path from any point of an elliptic region to its eccentric point is a straight line. Moreover, from Prop. 2, we know that the eccentric point p_e of any point $p_l \in S_l$ is in S_r. Thus, for computing the eccentricity of p_l, we just have to find the point p_{sa} of the small axis, such that the direction (p_l, p_{sa}) is the same as the direction stored in p_{sa} (see Fig. 2).

Eccentric lines through the bigger axis. In the previous section, we have shown that it is possible to decompose an ellipse S along its smaller axis to efficiently compute the eccentricity ECC_S. This is not the case when decomposing S into S_u and S_d along the bigger axis $[-a, a]$ because

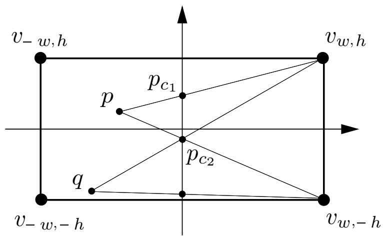

Fig. 4. Eccentric paths inside a rectangle

- the eccentric points p_e of any point $p_{ba} \in [-a, a]$ are either $(-a, 0)$ or $(a, 0)$, which is obviously not helpfull for deducing ECC_S;
- even if we associate to each point $p_{ba} \in [-a, a]$ the point $p' \in \partial S_u$ s.t. (p, p') is orthogonal to the tangent at p', $\exists p \in S_d$ with at least two points p'_1 and p'_2 in ∂S_u s.t. (p, p'_1) and (p, p'_2) are orthogonal to the tangent at p'_1 respectively p'_2. So we cannot make a one to one mapping for a direct computation of $ECC_S(p)$ as $d(p, p'_1)$ and $d(p, p'2)$ have to be compared. See Fig. 3.

Circle. In the special case where $a = b$, the shape S becomes a circle. Let $r = a = b$ be the radius and O the center. For all $p \in \partial S$ the line orthogonal to the tangent at p contains the center O. Thus all the eccentric paths go through the center O and the eccentricity of a point $p(x, y) \in S$ is $ECC_S(p) = \sqrt{x^2 + y^2} + r$. Note that all the points of ∂S are eccentric points.

3.2 Rectangle

Even though the rectangle is a simple case, studying it's decomposition in the context of eccentricity is of importance. Compared to the ellipse, a one to one association between points on the cut and eccentric paths cannot be made. In the case of the rectangle, two eccentric point candidates exist for each point on the cut.

Let S be a rectangle with side lengths $2w$ and $2h$ (see Fig. 4). The four corners of the rectangle $v_{-w,-h}$, $v_{-w,h}$, $v_{w,-h}$, $v_{w,h}$ make up the set of eccentric points of S.

The rectangle S can be decomposed in two subparts S_l and S_r (see Fig. 4), along the cut $C = [(0, -h), (0, h)]$. The corners $v_{-w,-h}$, $v_{-w,h}$ respectively $v_{w,-h}$, $v_{w,h}$ are the eccentric points of all the points $p \in C$.

The main difference compared to the ellipse, is that in the case of the rectangle we cannot associate to each point of C a single pair made of a direction and distance, because eccentric paths with more than one orientation can pass though the same point of the cut (see p_{c_2} in Fig. 4). To solve this, to each point of C we associate two pairs of distances and directions, connecting it to the corners $v_{-w,-h}$, $v_{-w,h}$ respectively $v_{w,-h}$, $v_{w,h}$. Thus, for any $p \in S_l, \exists p_{c_1}, p_{c_2} \in C$ s.t. $p_{c_1} \in (p, v_{w,h})$ and $p_{c_2} \in (p, v_{w,-h})$. Then

$$ECC_S(p) = \max(d(p, p_{c_1}) + ECC_S(p_{c_1}), d(p, p_{c_2}) + ECC_S(p_{c_2})).$$

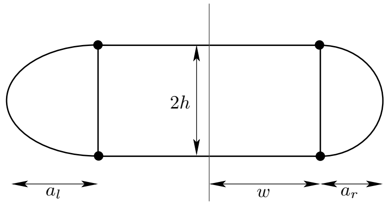

Fig. 5. Elongated shape formed of two half ellipses and a rectangle

3.3 Elongated Shape

Let S be an elongated shape obtained by gluing two oppsite sides of a rectangle R with two halves of ellipses E_l and E_r. Let us assume that: the width of R is $2w$ and its height is $2h$; E_l is the half of the ellipse defined by the 2 parameters a_l and h; E_r is the half of the ellipse defined by the 2 parameters (a_r, h) (See Fig. 5).

Symmetric Shape S: Let's assume that $a = a_l = a_r$ and decompose S along the cut $C = [(0, -h), (0, h)]$.

If $a \geqslant h$, we can extend Prop. 1 from Section 3.1 to elongated shapes. Thus, for any point $p \in S_l$, its eccentric point p_e is in S_r and is orthogonal to the ellipse tangent at p_e. Moreover, we can deduce that p_e is the unique eccentric point of p i.e. $ep(p) = \{p\}$.

Like in Section 3.1, we can compute the eccentricity and the eccentric path orientation for all the points of C separately in S_l and S_r and for all $p \in S_l$ compute the eccentricity $ECC_S(p) = d(p, p_c) + ECC_{S_r}(p_c)$ where $p_c \in C$ and (p, p_c) has the same direction as the eccentric path of p_c in S_r. These results can directly be extended for the points of S_r. To find the eccentricity and eccentric path orientation for C in S_l and S_r one can decompose S_l and S_r in the half-ellipse together with half the rectangle R, and directly use the formulas provided in Section 3.1.

Note that if $a = h$ then E_l and E_r are half circles and all the eccentric paths will go through their centers.

If $r > h$, the ellipses E_l and E_r correspond to ellipses that have been cut along their bigger axis (see Section 3.1). In this case, as mentioned in Section 3.1, there exist some points $p \in S_l$ which have more than one associated points $\{e_1, e_2, \ldots, e_k\} \in S_r$, such that (p, e_i) is orthogonal to the ellipse tangent at e_i.

Thus, we cannot associate each point in S_l with a single direction and distance based only on the orthogonality with the ellipse tangent. We need to take the maximum of the distances.

Nonsymmetric Shape S: If $a_l \neq a_r$, S is no longer symmetric and it cannot be decomposed by a line segment s.t. for any point $p \in S$ all its eccentric points are contained in the part not containing p. An easy way to overcome this problem is to take for each point, the maximum between the eccentricity computed separately on the part containing the point and the one obtained by using the decomposition.

4 Outlook and Conclusion

In this paper we have studied top-down decomposition of basic shapes in order to speed up the computation of the eccentricity transform. Some partitions proved to be better suited than others. We showed that these shapes can be decomposed for a more efficient computation and also derived some properties that could be applied for more general shapes. In particular, we provide a study regarding possible decompositions and their properties for the ellipse, the rectangle and a class of elongated shapes. In the future we plan to extend this study to any 2D shape followed by a study for 3D and nD shapes.

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