Nonlinear Approximation of Spatiotemporal Data Using Diffusion Wavelets

Marie Wild

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Motivation of this talk

• Wavelet based methods:
proven to be successful in signal and image analysis
main applications: edge-preserving smoothing and denoising of functions in $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$

• Recent concept of diffusion wavelets (Coifman et al.):
construction of wavelet bases for functions defined on other than $\mathbb{R}^n$, such as certain domains, manifolds and **graphs**.

• In this talk: study the use of classical wavelet algorithms in a graph based setting:

Input data: an image sequence, regarded as $2d+$ time data set
Model: the whole image sequence as a weighted graph
Output: Compressed data set, structure-preserving smoothing
Further goal: Spatiotemporal segmentation via diffusion wavelets

Outline of this talk

1. Wavelets and Multiresolution Analysis on $\mathbb{R}$
   (a) Orthonormal Wavelet Bases
   (b) Nonlinear Wavelet Approximation
2. Wavelet Analysis on a Graph
3. Nonlinear Approximation of Spatiotemporal Data Using Diffusion Wavelets
4. Conclusion and Outlook Towards Spatiotemporal Segmentation

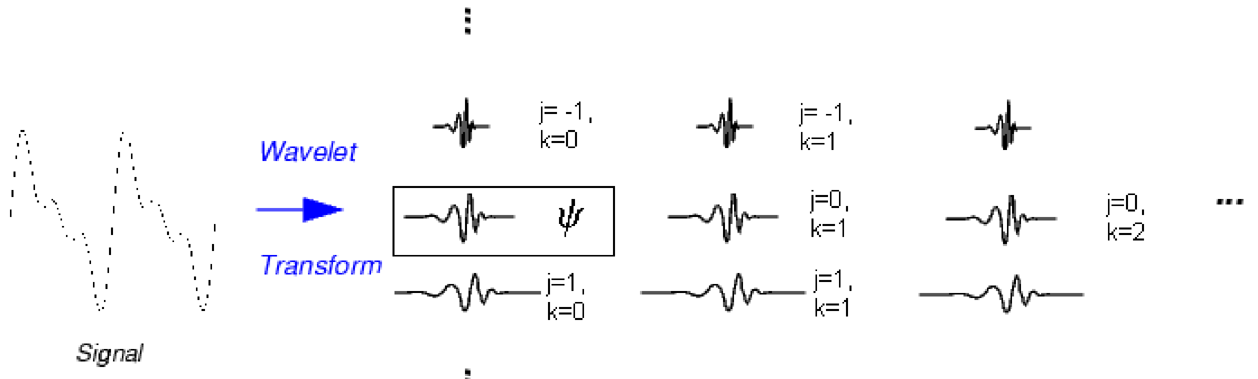
1. Wavelets and Multiresolution Analysis on $\mathbb{R}$

(a) Orthonormal Wavelet Bases

- Wavelet transform:

Decomposition of a function (1d signal or 2d image) into a series constituent of localized waves $(\psi_{j,l})_{j,l\in\mathbb{Z}}$

$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,l}\rangle \psi_{j,l}$, where $\psi_{j,l}$ are dilated (squeezed/stretched) and translated versions of a mother wavelet $\psi \in L^2(\mathbb{R})$.



Constituent wavelets of different scales and positions

In applications:

Calculation of coefficients via fast wavelet transform, using a cascade of filters

$f \xrightarrow{\text{FWT}} (d_{j,k})_{j,k\in\mathbb{Z}}$,

$$\begin{array}{ccccccccccccc}
f = a_0 & \longrightarrow & a_1 & \longrightarrow & a_2 & \longrightarrow & \dots & \longrightarrow & a_{j-1} & \longrightarrow & a_j \\
& \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
& & d_1 & & d_2 & & & & d_{j-1} & & d_j
\end{array}$$

where each horizontal arrow represents the same filtering and subsampling step $a_{j+1} = \downarrow_2 (a_j * g)$, and similarly, $d_{j+1} = \downarrow_2 (a_j * h)$, $\quad g, h$ CMF.

$(a_{j,l})_l$ approximation coefficients at scale (resolution) j,
$(d_{j,l})_l = (\langle f, \psi_{j,l}\rangle)_l$ wavelet coefficients at scale j.

(a) Nonlinear Wavelet Approximation

$(\psi_{j,l})_{j\geq 1, l\in\mathbb{Z}}$ ONB $\Longrightarrow$ for $f \in L^2(\mathbb{R})$,

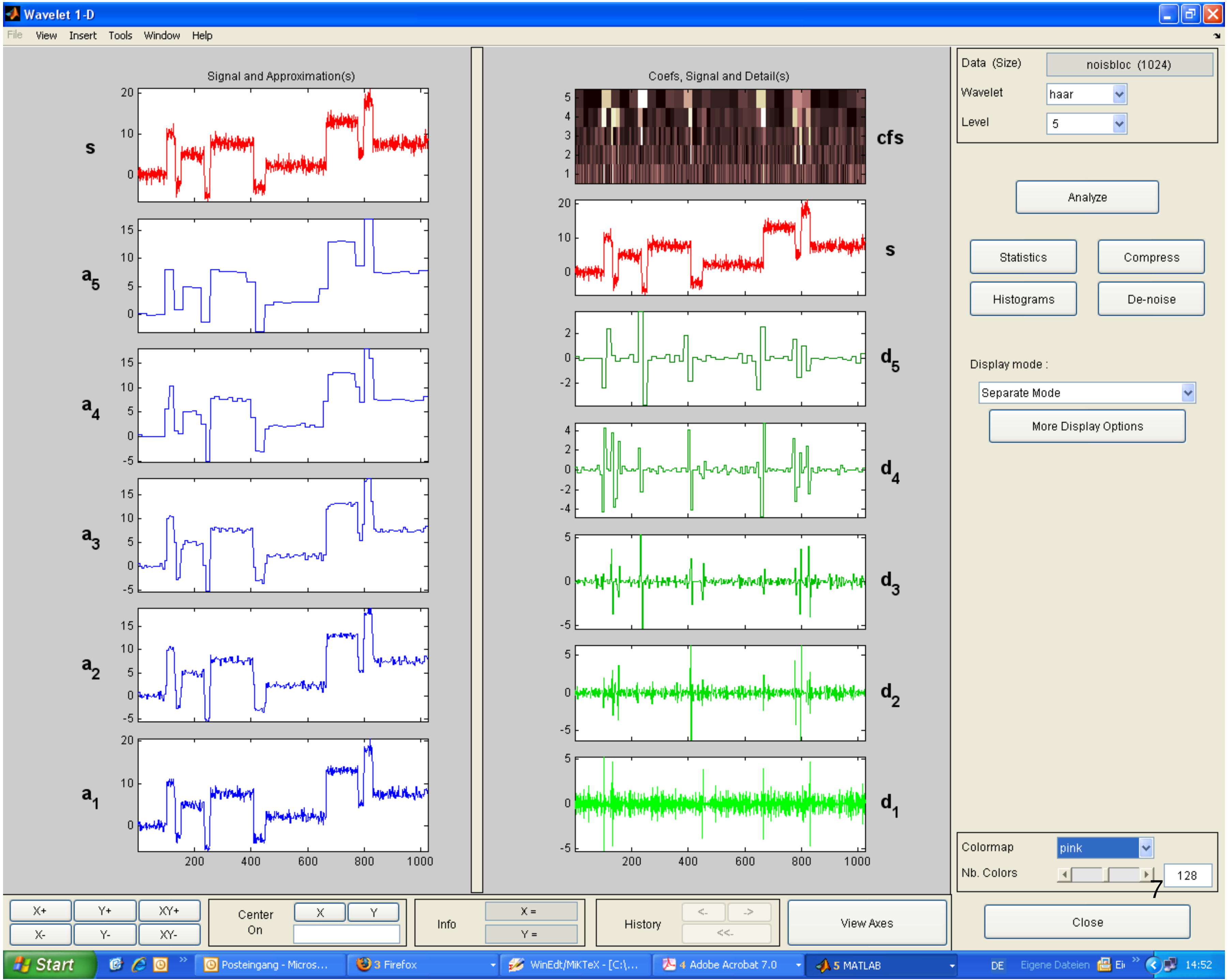
$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{j,l} |\langle f, \psi_{j,l}\rangle|^2.$$

$\Longrightarrow$ all the information of f maintained in the sequence of coefficients, salient information is reflected in the largest coefficients

$\Longrightarrow$ efficient approximation using only the N largest coefficients for reconstruction, realized by thresholding on the coefficients

$\Longrightarrow$ discontinuity preserving smoothing,

large theory relating the error of approximation to the function's properties


Wavelet 1-D
File View Insert Tools Window Help
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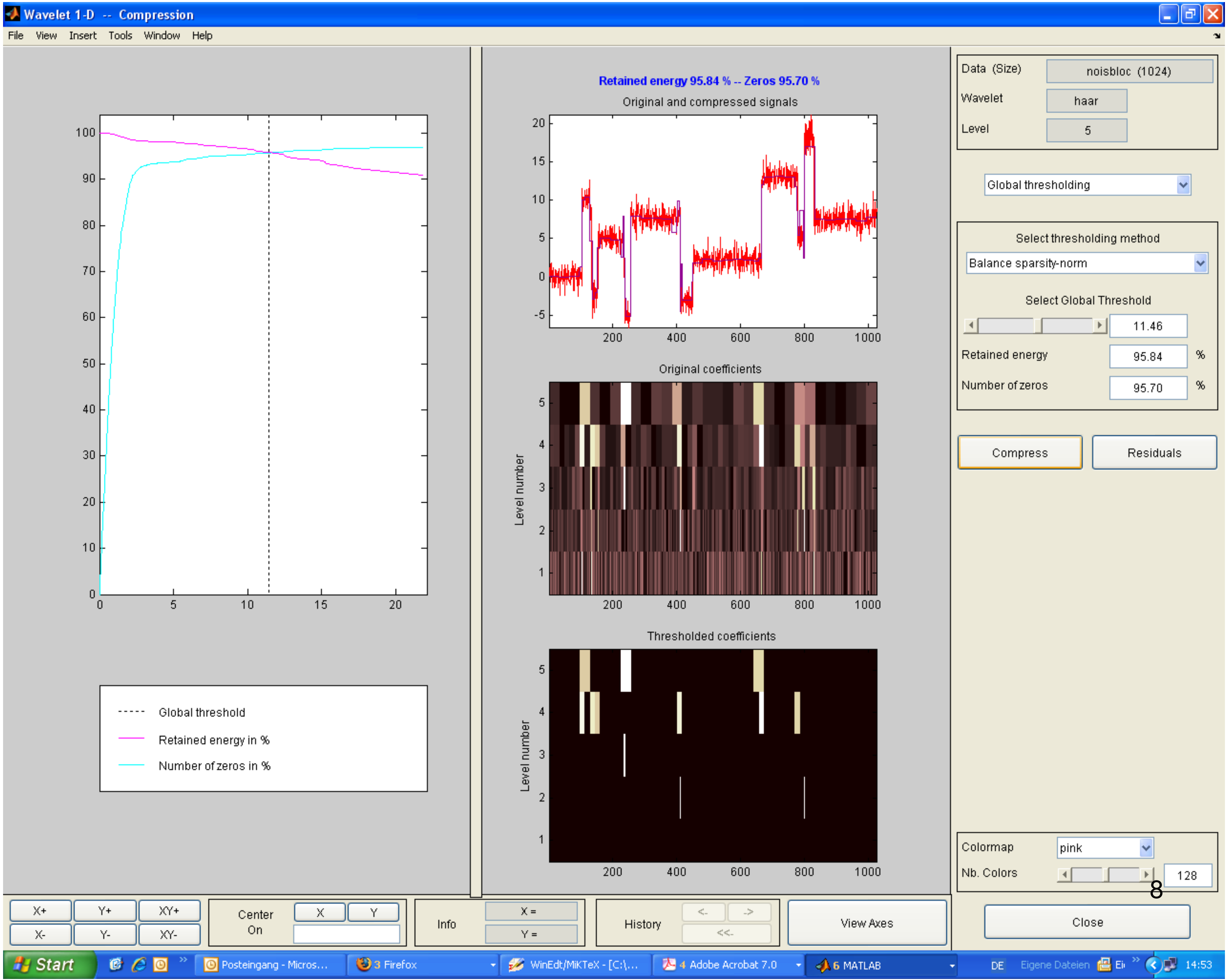
Wavelet 1-D -- Compression
File View Insert Tools Window Help
Global threshold
Retained energy in %
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Retained energy 95.84 % -- Zeros 95.70 %
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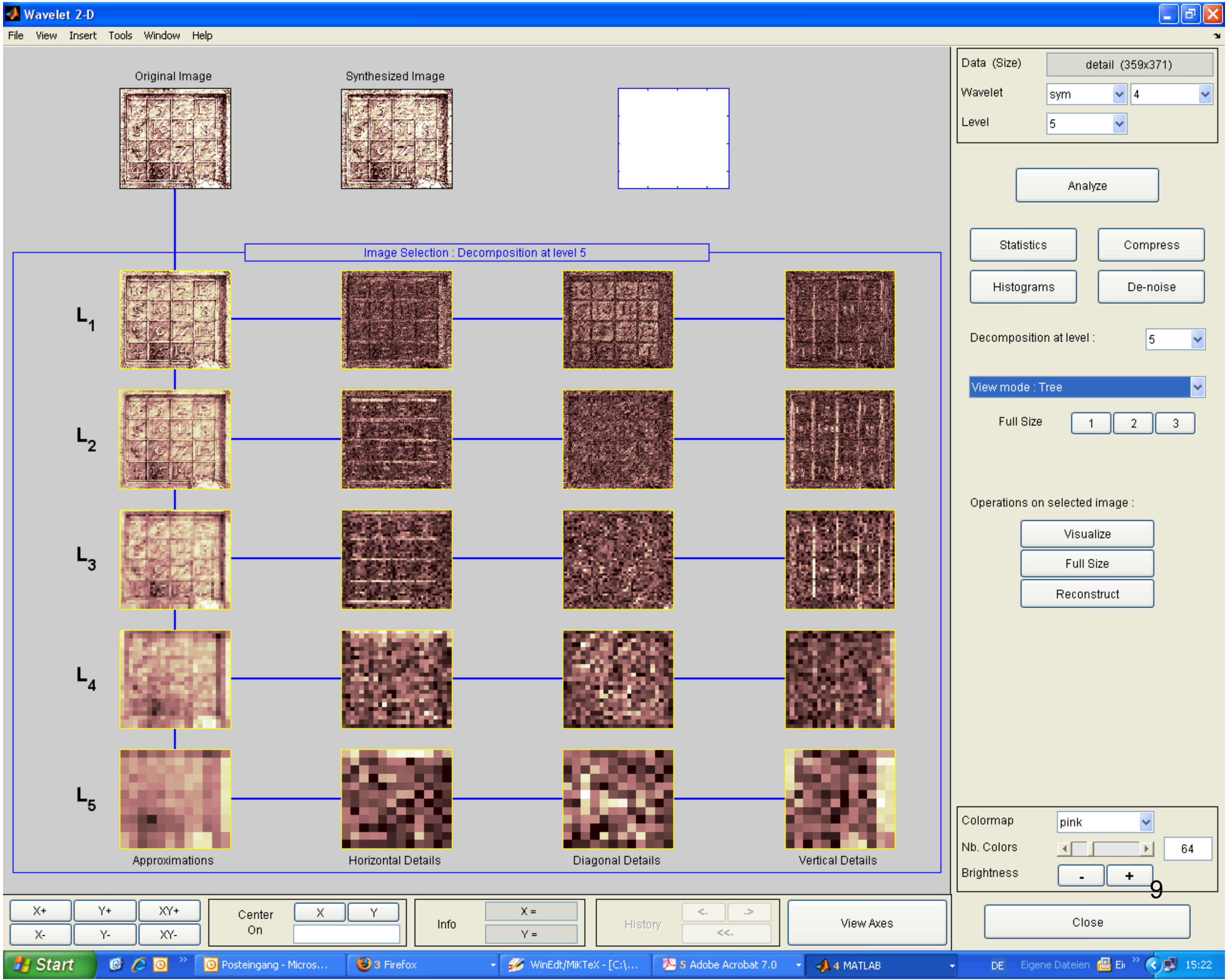
Wavelet 2-D
File View Insert Tools Window Help
Original Image
Synthesized Image
Image Selection : Decomposition at level 5
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Horizontal Details
Diagonal Details
Vertical Details
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Histograms
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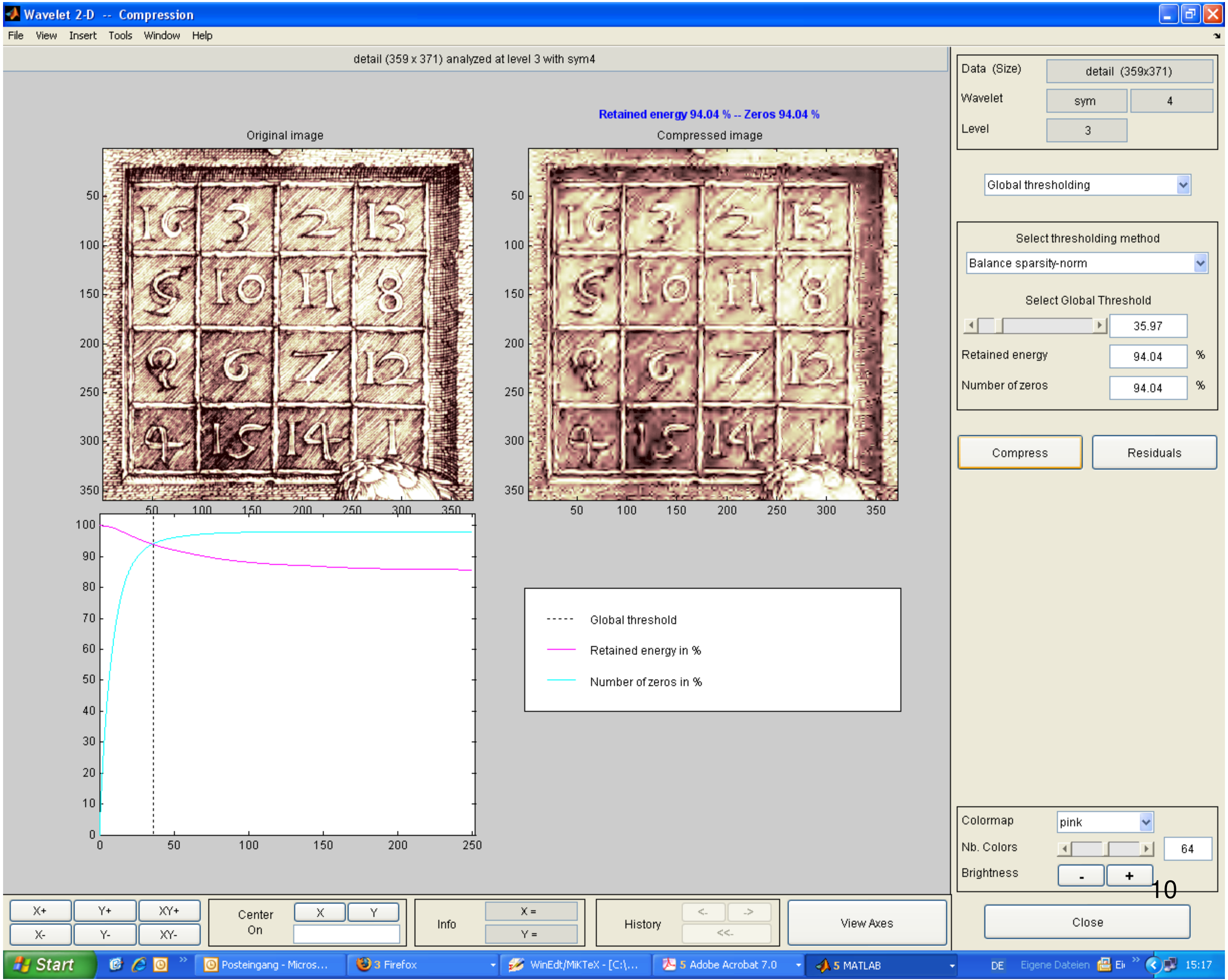
Wavelet 2-D -- Compression
File View Insert Tools Window Help
detail (359 x 371) analyzed at level 3 with sym4
Retained energy 94.04 % -- Zeros 94.04 %
Original image
Compressed image
Global threshold
Retained energy in %
Number of zeros in %
Data (Size)
detail (359x371)
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Global thresholding
Select thresholding method
Balance sparsity-norm
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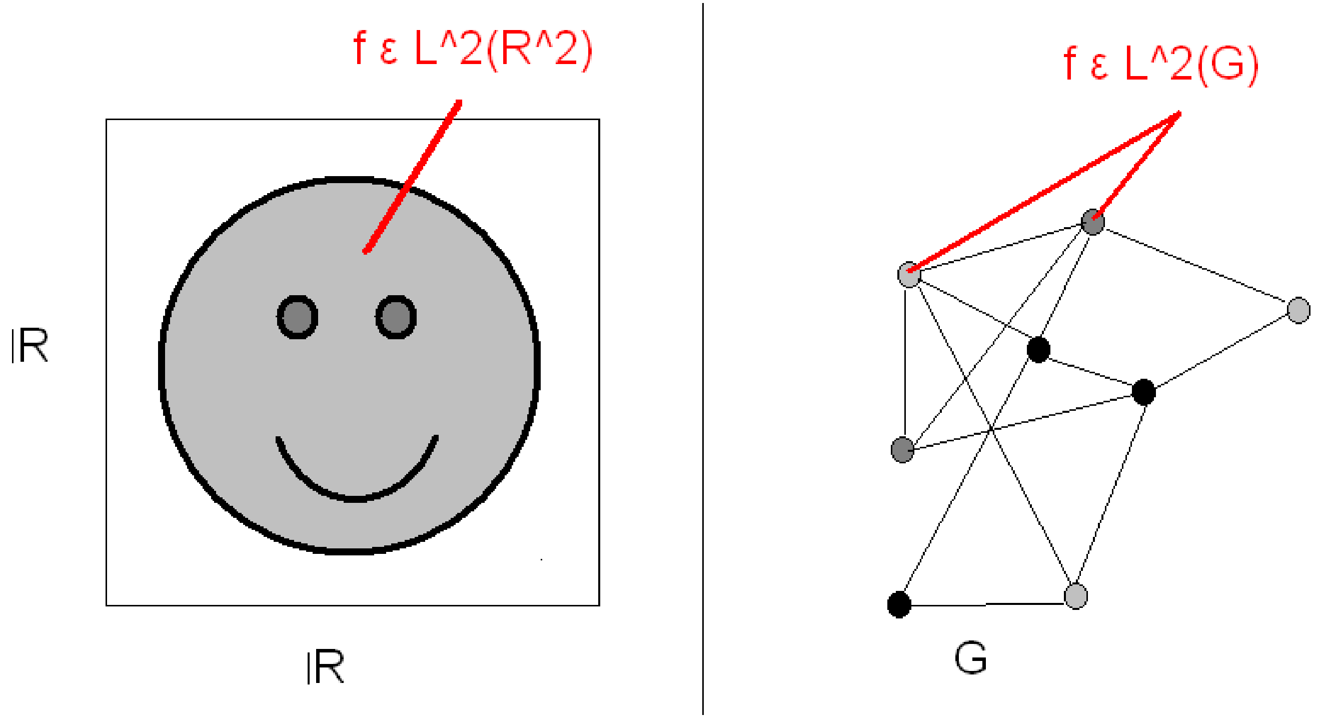
Close
Center On
Info
History
View Axes

2. Wavelet Analysis on a Graph

- How to extend wavelet analysis on a graph?:



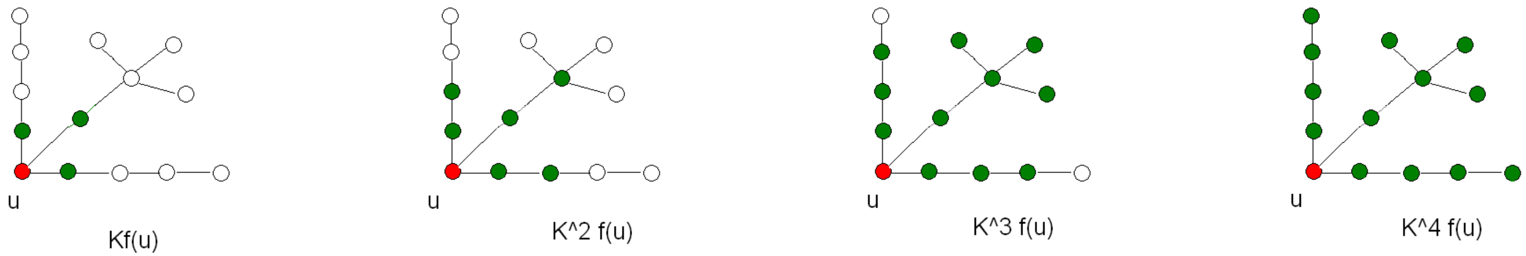
Dilations (stretching and squeezing) like on $\mathbb{R}$ cannot be used for scaling
$\Rightarrow$ use spectral methods to define a multiresolution analysis

FAN R.K. CHUNG: **SPECTRAL GRAPH THEORY**, CMBS-AMS, NO. 92, 1997

Diffusion operator K:

averaging operator acting on functions f on G:

$$Kf(u) = \frac{1}{\sqrt{d_u}} \sum_{v, v \sim u} \left(\frac{f(v)}{\sqrt{d_v}} - \frac{f(u)}{\sqrt{d_u}} \right) W(u,v)$$



Diffusion Wavelets

R.R.Coifman, S. Lafon, A.B. Lee, M. Maggioni, B. Nadler, F. Warner and S.W. Zucker: **Geometric diffusions as a tool for harmonic analysis and structure definition of data**, Proc. Nat. Ac. of Sc., Vol 102(21), May 2005
R.R. Coifman, M. Maggioni: **Diffusion Wavelets**, ACHA, Vol. 21(1), July 2006

Diffusion operator K as dilation operator acting on functions on $L^2(G)$

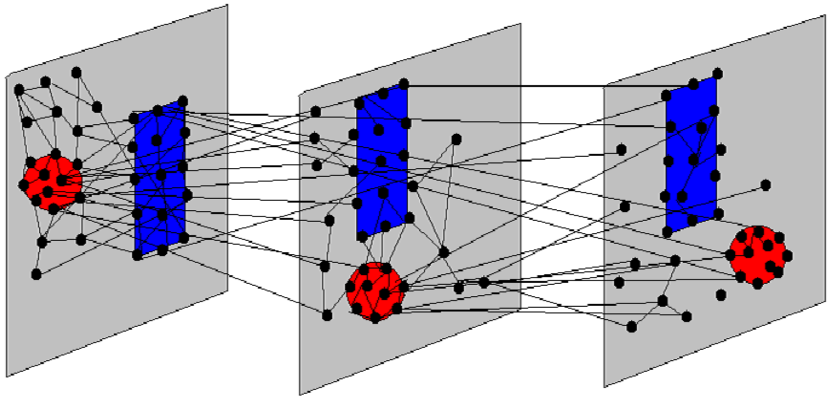
$\Rightarrow$ define multiresolution structure via dyadic powers of K: K^{2^j}, $j \geq 0$

Recipe: Divide the spectrum (eigenvalues) of K into different 'frequency bands' and find orthonormal bases for the spaces spanned by the corresponding eigenvectors

$\Rightarrow$ Multiresolution Analysis and ONB for functions on G

3. Nonlinear Approximation of Spatiotemporal Data Using Diffusion Wavelets

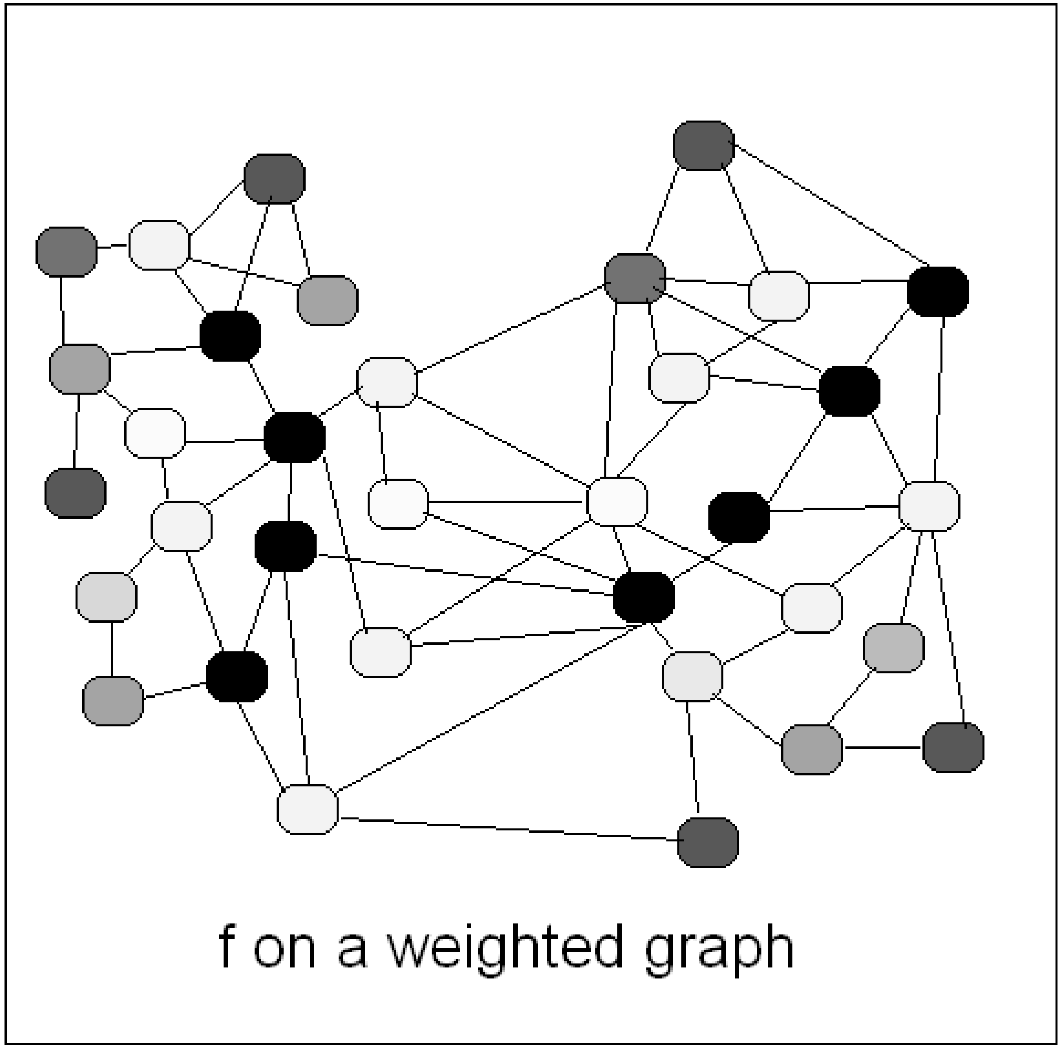
• build a weighted graph G from the $3d$ image data



vertices: whole set of pixels, or due to complexity considerations: a subset (downsampled version of the sequences/ filtering/feature point selection)
edges and weights: difference of intensities, distance in space, feature point properties, information from a *motion prediction* or a combination of the above
function f on G: additional attributes on the vertices, for the 'pure' graph $f \equiv 1$.

$\Longrightarrow$ Encode local similarities in G

Result: abstract graph G from the data, $f \in L^2(G)$



- Build diffusion wavelet basis $(\psi_{j,l})_{j \geq 0, l \in \mathbb{Z}}$ on G

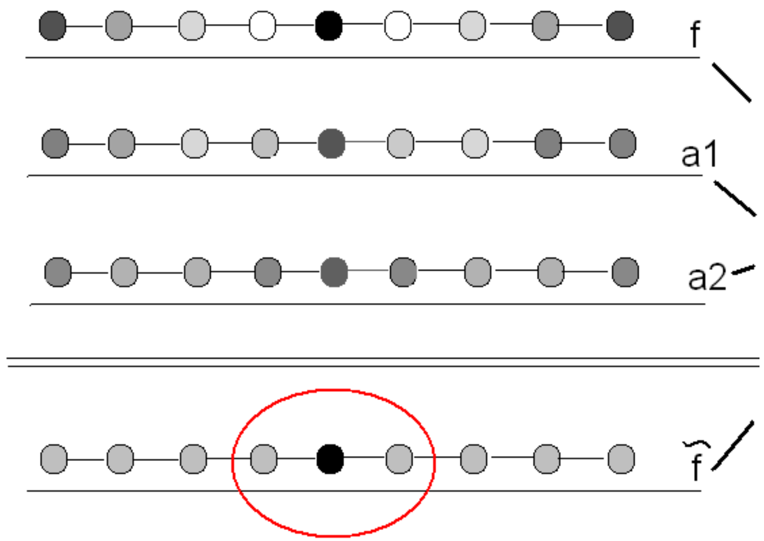
- Calculate wavelet coefficients $(\langle f, \psi_{j,l} \rangle)$

$\Longrightarrow$ **Result**:
sequence of coefficients like in classical wavelet transform, **but**:
information encoded now defers to structural similarity of the data instead of properties on the fixed grid $\mathbb{R}$

- Approximations on different scales available

- **Structure-preserving** compression via thresholding on the coefficients

Example



function on a graph (edge weights as differences of grey values)

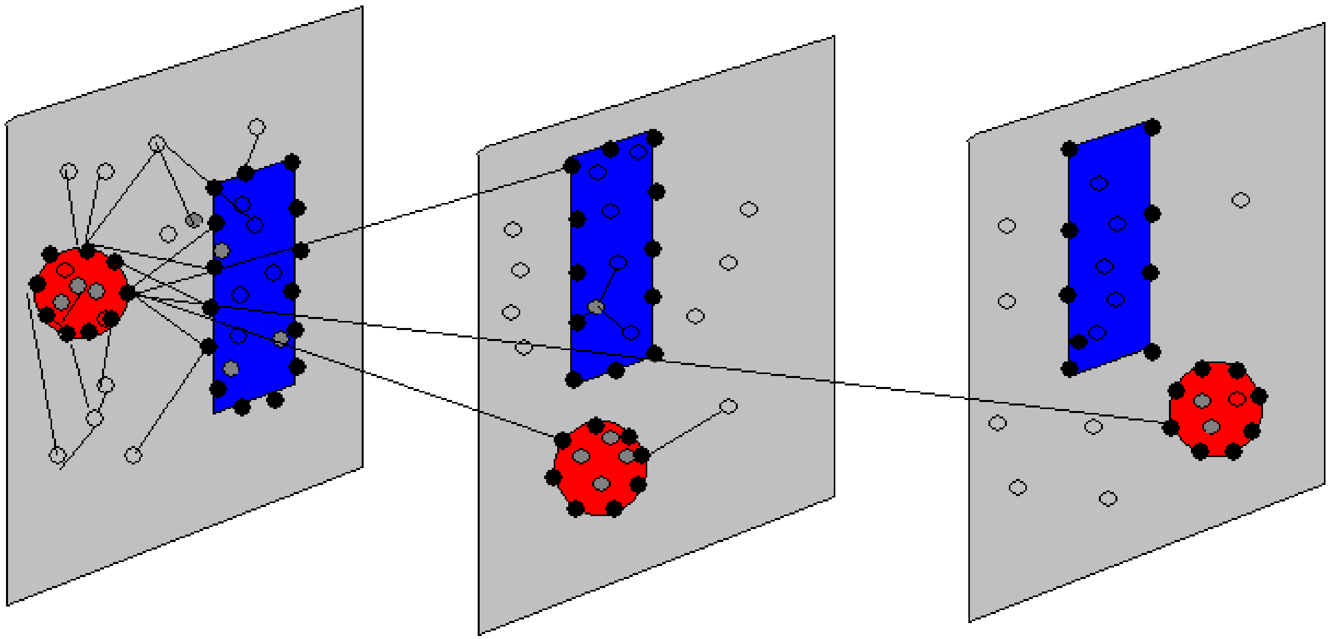
wavelet approximations at different scales

approximation via largest coefficients (thresholding)

• Function is **smoothed** where local changes of the weights are small, where **'discontinuities'** (large weight changes) are **preserved**

• Remember: weights encode local similarities (intensities, motion properties)
$\Rightarrow$ structure-preserving compression

Structural approximation of the image sequence



4. Conclusion and Outlook Towards Spatiotemporal Segmentation with Diffusion Wavelets

• Diffusion wavelet bases lead to a true multiscale decomposition on a graph
$\Rightarrow$ Opens the door for a multitude of (graph based) CV tasks

Presented here as a first step:

Algorithm for structure-preserving compression using diffusion wavelets

• derived from classical wavelet methods now lifted to a graph-based setting
$\Rightarrow$ theory relating properties of the function to the approximation error also available in this setting

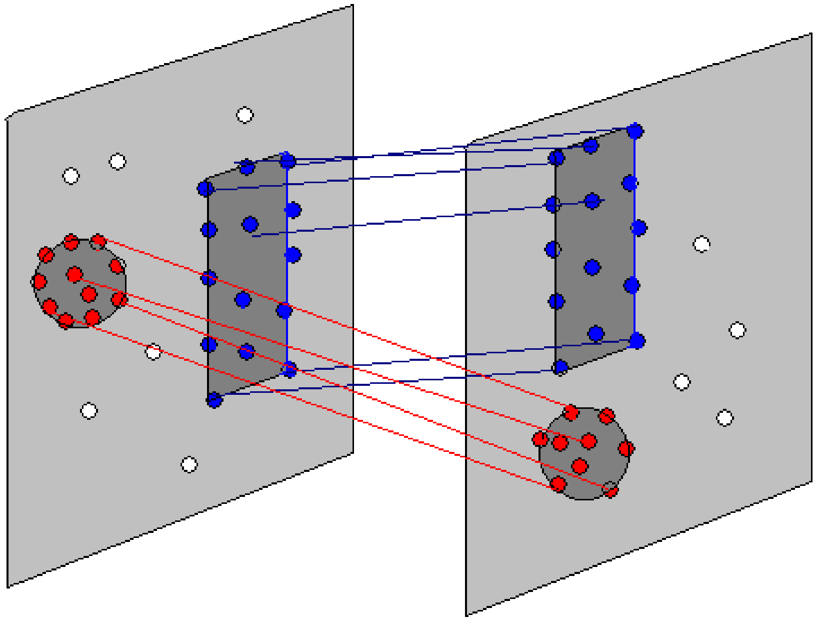
• data smoothed by the approximation, abrupt changes in the edge weights (which can describe object borders in single images or the main direction of movement) are preserved

$\Rightarrow$ Now working on implementation and experiments

Next step:

$\Rightarrow$ Future work: spatiotemporal segmentation using diffusion wavelets (labelling of vertices via a HMM on the coefficients)

Outlook: Spatiotemporal Structural Segmentation



Segmentation by **labelling of graph vertices** (different colors),
Grouping by local similarities (intensities/motion profile/combined) via classifying diffusion wavelet coefficients across scales